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Quantum Gravity

A Search for Theory
of Everything

DeepScience

Quantum Gravity: A Search for Theory of Everything

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Prepare

This book presents a coherent and conceptually motivated exploration of fundamental physics, tracing the intellectual journey from the early geocentric worldview of Claudius Ptolemy to modern unification efforts such as M-theory. Rather than offering a purely technical treatment, it emphasizes the evolution of ideas that have shaped our understanding of the universe, highlighting how each scientific revolution in physics has historically arisen from the inability of existing theoretical frameworks to account for newly observed phenomena or internal conceptual inconsistencies. The scope of this work spans classical mechanics, relativity, quantum theory, and modern attempts at grand unification. Beginning with the transformation from Aristotelian cosmology to the heliocentric understandings of Nicolaus Copernicus and the dynamical laws of Issac Newton, the discussion progresses through the conceptual breakthroughs of Albert Einstein in relativity and the probabilistic foundations of quantum mechanics developed by pioneers such as Max Plank and Neils Bohr. The story concludes in modern theoretical frameworks including string theory and higher dimensional models that seek to reconcile gravity with quantum field theory. The target readership includes advanced undergraduate and graduate students in physics, interdisciplinary researchers, and intellectually curious readers who seek conceptual clarity rather than dense mathematical derivations. While mathematical structure is acknowledged as essential, the primary focus remains on physical meaning, logical development, and philosophical implications. The aim is to make the deep continuity of physical thought accessible without losing academic consistency. This book distinguishes from existing texts is its importance on conceptual continuity and critical reflection. Instead of presenting physical theories as isolated achievements it frames them as stages in an ongoing scientific dialogue each successful within its domain yet incomplete in a deeper sense. By foreground processing the motivations, conceptual tensions, and unresolved paradoxes that drive theoretical progress, this book clarifies why a complete " Theory of everything" remains incomplete and what intellectual transformation such a theory would require. Ultimately, this book invites readers to view physics not just as a collection of equations, but as a dynamic and evolving human effort one that continually reshapes our understanding of reality and our place within the cosmos.

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Chapter # 01

Foundation of Classical Gravitation

In incident time peoples observed the sky with the naked eyes. They assumed that the earth was to be stationary at the center of the universe, while the sun, moon, planets, and stars revolving around it, this concept was called the geocentric model of astronomy is seen as in figure 1.1. This concept was deeply rooted in Aristotelian natural philosophy, which stated that heavy bodies naturally moved toward the center of the cosmos and therefore accumulated at earth's position. Celestial bodies were thought to be composed of a distinct perfect substance and to move according to fundamentally different principles than those governing terrestrial objects. The most advanced mathematical formulation of the geocentric model was provided by Claudius Ptolemy in the second century CE. In order to resolve uniform circular motion with the observed irregularities in planetary paths, Ptolemy introduced a complex geometric system involving different epicycles and equants. Each planet moves on a circular path whose center followed a large circle around the earth, with angular motion measured relative to a displaced point rather than the center itself. Although this system required a physical explanation for why celestial bodies should move in such a manner, it achieved considerable success in predicting planetary positions. As a results, the geocentric model continued not because of its simplicity, but its empirical capability within the limits of available observations.
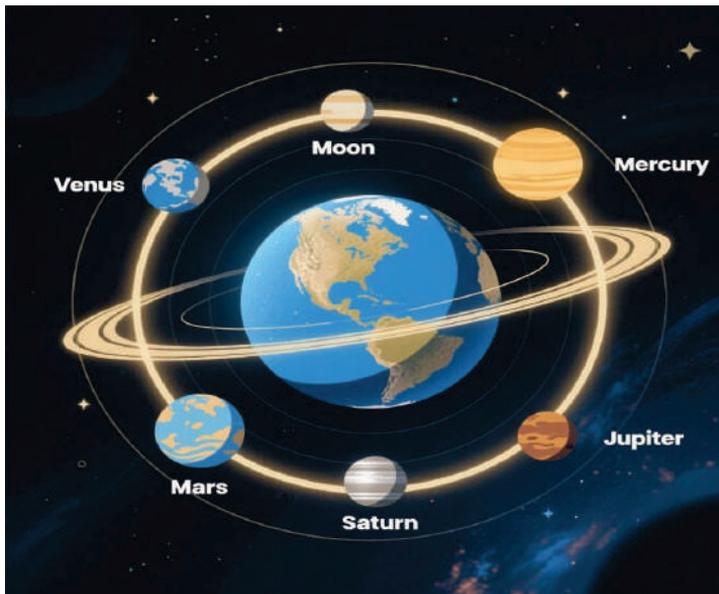

Figure 1.1: Geocentric model of the astronomy where earth at rest and at the center of all moving planets.

Regardless its successes, the geocentric model faced increasing strain as observational precision improved. The growing complexity required to maintain agreement with data raised questions about its assumptions. These difficulties motivated the search for alternative arrangement of the cosmos, leading to the renewal of heliocentric ideas during the renaissance. The heliocentric model of astronomy proposed that the sun, rather than earth, occupied the central position in the planetary system, with earth itself moving both rotationally and translationally is presented in figure 1.2. Nicolaus Copernicus presented the most influential early modern formulation of heliocentrism in the sixteen centuries. By assigning daily celestial motion to earth's rotation and annual changes to its revolution around the sun, Copernicus provided a natural explanation for phenomena such as retrograde planetary motion. The apparent reversal of a planet's direction was no longer treated as an intrinsic irregularity, but as a consequence of relative motion between earth and other planets. While Copernicus retained circular orbits and continued to employ epicycles, his model offered a more coherent ordering of the planets and a unified explanation for their observed motion. Initially, the heliocentric model did not significantly outperform the Ptolemaic system in predictive accuracy and run into strong philosophical and theological resistance. Its deeper significance unprofessional in its conceptual reorientation of the cosmos and its implicit challenge to the privileged status of earth. The acceptance of heliocentrism required not only mathematical reconstructing, but also significant empirical evidence and a new understanding of motion.

Figure 1.2: Heliocentric model: Sun is the center of all planets and all planets moving around it.

Such evidence emerged through the work of Galileo Galilei, whose contributions marked a turning point in the history of science. By employing the telescope as a systematic observational instrument, Galileo exposed structures of the heavens that opposed fundamental assumptions of the geocentric model. His discovery of moons orbiting Jupiter demonstrated that not all celestial

motion was centered on the earth, while the observed phases of Venus could only be explained by its orbit around the sun is seen as in figure 1.3. Observations of lunar surface irregularities and sunspots further undetermined the Aristotelian distinction between perfect celestial realms and imperfect terrestrial ones. Galileo's influence extended beyond observational astronomy to the foundation of mechanics. Through controlled experiments and quantitative analysis, he demonstrates that bodies in free fall experience uniform acceleration and that motion continuous in the absence of resisting forces. These results directly challenged Aristotelian dynamics, which held that continuous motion required continuous application of force. Galileo's formulation of inertial motion and his insistence on mathematical description transformed natural Philosophy into a discipline grounded in empirical testing and quantitative laws.

Figure 1.3: Galileo's observation of celestial bodies.

The unification of these emerging ideas was accomplished by Isaac Newton in the late seventeenth century. Building directly upon Galileo's analysis of motion, newton formulated a complete theoretical framework capable of explaining both terrestrial and celestial phenomena with in the same set of physical principles. In this formulation of the laws of motion, newton precisely defined the relationship between force, mass, and acceleration providing a general mathematical framework for mechanics. These laws established inertia, quantified acceleration as a result of applied forces, and formalized the concept of action and reaction. The main work of the newton is universal law of gravitation, is seen as in figure 1.4, which stated that every mass in the universe

attracts every other mass with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. The mathematical form given by Eq. (1.1),

$$F = G \frac{m_1 m_2}{r^2}$$ (1.1)

Where m1 and m2 masses of the objects, r is the distance between them, and G is the universal gravitational constant. This principle applied universally, governing the fall of objects near earth as well as the motion of the moon, planets and comets. Newton's theory eliminated the traditional separation between celestial and terrestrial physics and replaced it with a single deterministic description of nature.
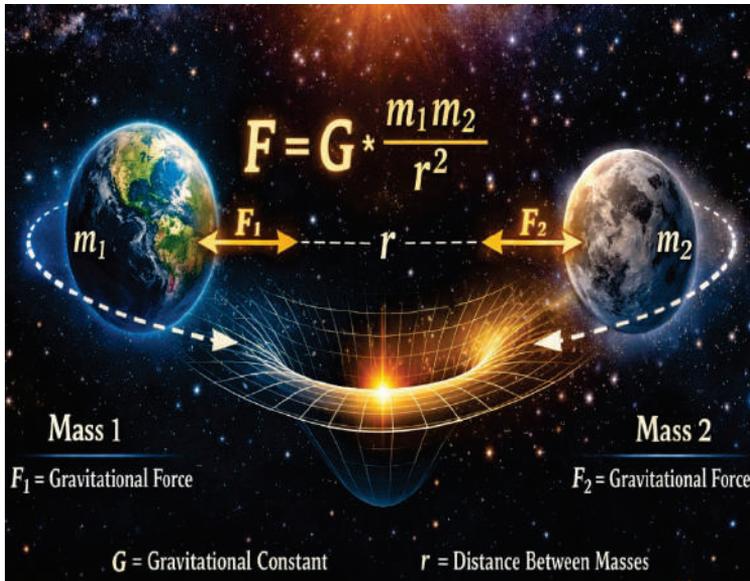
Fig 1.4: Represent universal law of gravitation where two masses attract each other by a force.

While newton provided the dynamic explanation for planetary motion, the detailed empirical regularities governing planetary orbits had been established earlier by Johannes Kepler. Kepler's working from the precise astronomical observations of Tycho Brahe, formulated the three laws of planetary motion that described the kinematics of planets without involving the underlying forces. His first law, stated that, planets move in elliptical orbits with the sun at one focus, known as law of ellipses is shown in figure 1.5 (a), replacing the long-standing assumptions of circular motion. The second law stated that, a line joining a planet to the sun sweeps out equal areas in equal intervals of time, known as law of areas is seen as in figure 1.5 (b) implying a variable orbital speed. Mathematical form of Kepler's second law,

$$\frac{dA}{dt} = \frac{L}{2m} = constant$$ (1.2)

Where dA/dt is Areal velocity, L angular momentum of the planets and m is mass of the palnets. The third law, known as law of harmonies is shown in figure 1.5 (c), established a precise mathematical relationship between a planet's orbital period and the size of its orbit, revealing a harmonic order within the planetary system. Mathematical form of Kepler's third law,

$$T^2 \propto a^3 \tag{1.3}$$

Where T is the orbital period of the planet and a is the semi major axis. Eq. (1.3) stated that square of a planet's orbital period (T^2) is proportional to the cube of the semi major axis (a^3), meaning farther planets take significantly longer to orbit the sun.
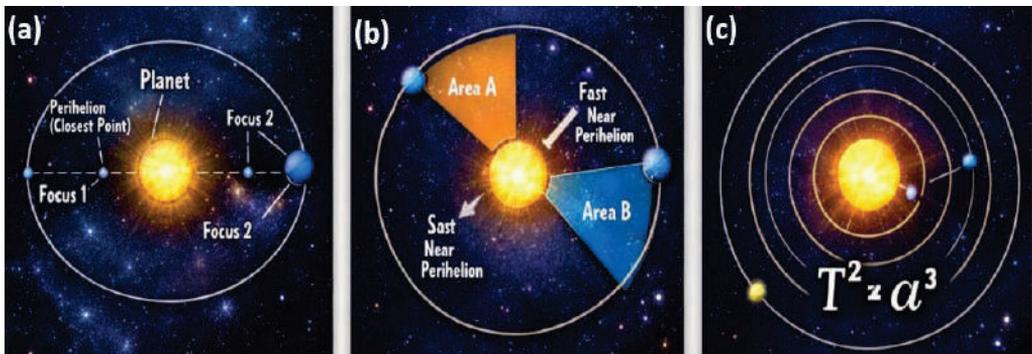
Figure 1.5: (a) The law of ellipses, (b) the law of equal areas while (c) the law of harmonies.

Kepler's laws represented a conclusive break from classical cosmology by leaving uniform circular motion in favor of empirically determined trajectories. However, they did not explain why planets move as they do. This explanation gap was closed by Newton, who demonstrated that Kepler's laws arise naturally from motion under an inverse square gravitational force. In this way, Kepler's empirical discoveries and Newton's theoretical work together placed the heliocentric model on a difficult physical foundation.

Chapter # 02

Electromagnetic Field Theory

The story of electricity begins in ancient times, with Greek observations that rubbing amber against cloth could attract light objects. These mysterious phenomena called elektron fascinated early thinkers but remained a curiosity without a quantitative understanding. It was not until the 17th and 18th centuries that electricity began to be studied systematically. William Gilbert distinguished between magnetic and electric effect, creating the term electric force for the attraction and repulsion of charged particles. The first major quantitative invention came with the work of Charles-Augustin de Coulomb in 1785. Using a torsion balance, Coulomb measured the force between two small charged spheres and discovered that it followed a simple law: the force between two-point charges q1 and q2, separated by a distance r, is proportional the product of magnitudes of charges and inversely proportional to the distance between them known as coulomb's law is seen as in figure 2.1, mathematically represented by Eq. (2.1),

$$F = k \frac{q_1 q_2}{r^2} \hat{r} \tag{2.1}$$

Where F is the coulomb's force, q1 and q2 is the magnitudes of charges, k is coulomb constant and $\hat{r}$ is a unit vector pointing from one charge to another. This law not only quantified the electric force but also directed to the concept of the electric field, a vector field that describes the influence of a charge exerts in the surrounding space. The electric field E, at a point is defined as the force per unit charge experienced by a small test charge, mathematically represented by Eq. (2.2),

$$E = \frac{F}{q} = k \frac{Q}{r^2} \hat{r} \tag{2.2}$$

This equation tells us that, the electric field originates radially from a point charge is inversely proportional to the distance r, and proportional to the coulomb's force.
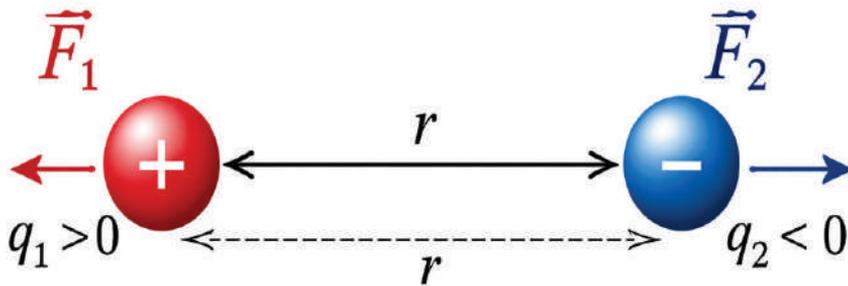
Figure 2.1: Representation of Coulomb's law where two charges attract or repel each other by coulomb's force .

For the understanding of electric field, gauss's law formulated in the 19th century, provides a powerful method to calculate electric fields, especially for symmetric contribution. It states that, the total flux through a closed surface is proportional to the charge enclosed, is seen as in figure 2.1. Mathematically represented by Eq. (2.3),

$$\oint E.\, dA = \frac{Q_{inclosed}}{\varepsilon_o} \tag{2.3}$$

Here dA, is an infinitesimal area element vector, $Q_{inclosed}$ is the total charge in the surface, and ε_o is the permittivity of free space. For example, Gauss's for a spherical charge distribution, directly reproduces coulomb's inverse square law, but it also simplifies calculations for infinite planes, cylinders, and other geometries where symmetry can be broken.
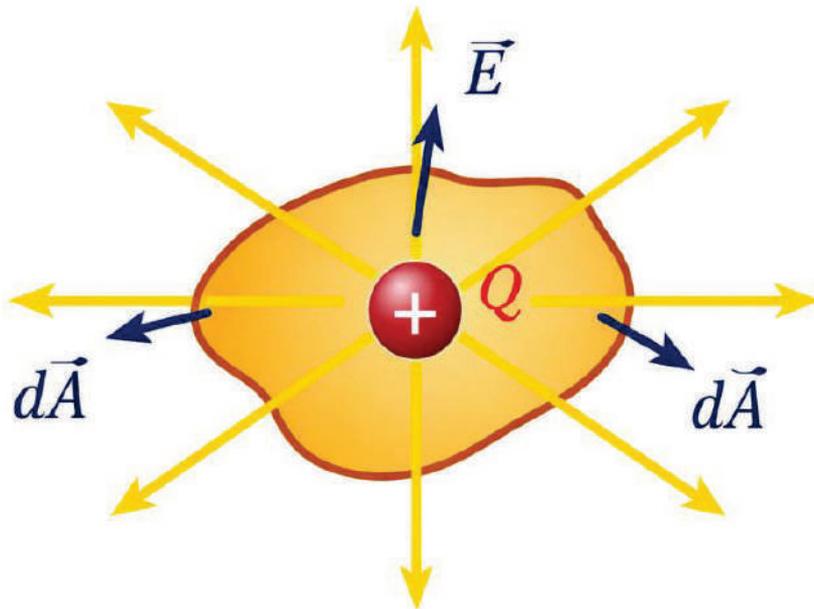
Figure 2.2: Graphical representation of gauss's law.

Electric field describes static charges, while electric current describes the motion of charges. Current I, defined as the rate of flow of charge, mathematical represented,

$$I = \frac{dQ}{dt} \tag{2.3}$$

Current measures in amperes, one ampere corresponds to one coulomb of charge passing through a conductor per second. In the 19th century the study of currents advanced to the discovery of the link between electricity and magnetism. Hans Orsted observed that a current carrying wire

deflected a nearby compass needle, demonstrating that moving charges produce magnetic effects. Andre-Marie Ampere formulated the quantitative relationship between currents and magnetic fields, as Ampere's law, is seen as in figure 2.3, which states that " current carrying wire produces magnetic field around it" mathematically can written as,

$$\oint B.dl = \mu_o I_{enclosed}$$ (2.4)

Where B, is magnetic field, dl, is an infinitesimal segment along a closed path, and μ_o is permeability of free space. Eq. (2.4) tells us that, the magnetic field B, around a closed loop is proportional to the electric current I, passing through that loop.
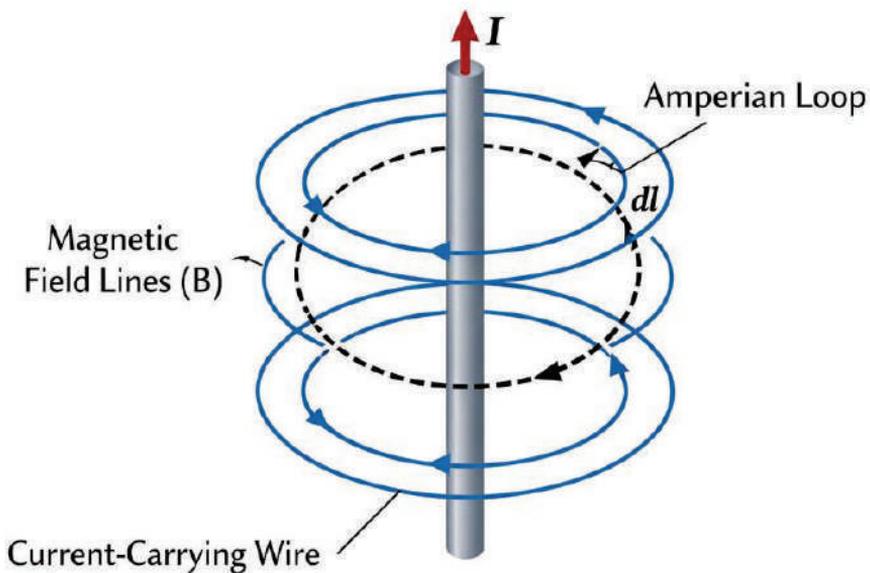
Figure 2.3: Ampere's law where current carrying wire produces magnetic field around it.

Simultaneously, Faraday law discovered electromagnetic induction, showing that a changing magnetic field induces an electric filed is seen as in figure 2.4. The induced electromotive force in a loop is given by Faraday law,

$$\varepsilon = -\frac{d\Phi_B}{dt}$$ (2.5)

Where Φ_B represents the magnetic flux through the loop. This principle underlies the operation of generators and transformers, forming the basis of modern electrical power systems. The practical exploitation of electricity advance to the development of direct current (DC) systems, supported by Thomas Edison, and alternating current (AC) systems, pioneered by Nikola Tesla and George Westinghouse. DC provides a steady flow of charges in one direction, while AC oscillates

periodically, allowing efficient transmission over long distances and easy voltage transformation. The so-called "War of currents" in the late 19th century ultimately favored AC, for large scale electricity distribution.
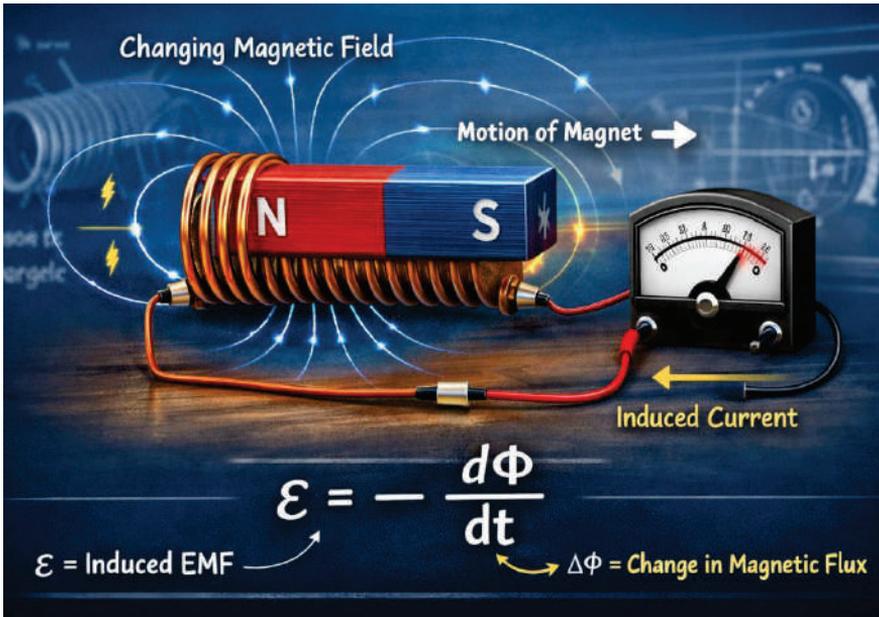
Figure 2.4: Faraday law of electromagnetic induction.

The theoretical unification of these phenomena came from James Clerk Maxwell who formulated four elegant equations linking electric and magnetic fields. Maxwell's equations can be expressed as,

$$\nabla . E = \frac{\rho}{\varepsilon_O} \tag{2.6}$$

$$\nabla . B = 0 \tag{2.7}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.8}$$

$$\nabla \times B = \mu_o J + \mu_o \varepsilon_o \frac{\partial E}{\partial t} \tag{2.9}$$

Eq. (2.6) represent gauss's law for electricity, Eq. (2.7) represent Gauss's law for magnetism is seen as in figure 2.5, and Eq. (2.8) represent Faraday's law of induction while Eq. (2.9) represent Ampere-Maxwell's law. Where ρ is the charge density, J is the current density, μ_o and ε_o are the permittivity and permeability of free space.
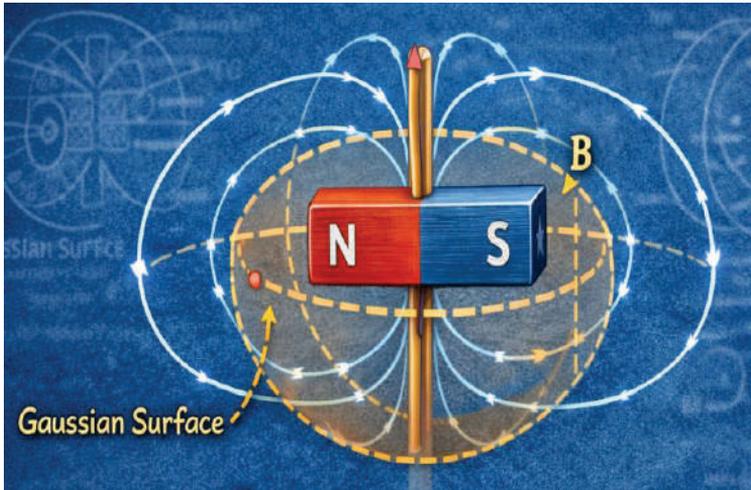
Figure 2.5: Gauss's law for magnetism.

Maxwell's insight revealed that changing electric field and magnetic field could sustain each other, propagating as waves at the speed of light,

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \tag{2.10}$$

This discovery not only unify electricity, magnetism and optics but also advance the foundation for radio, television and modern wireless communication is seen as in figure 2.6.
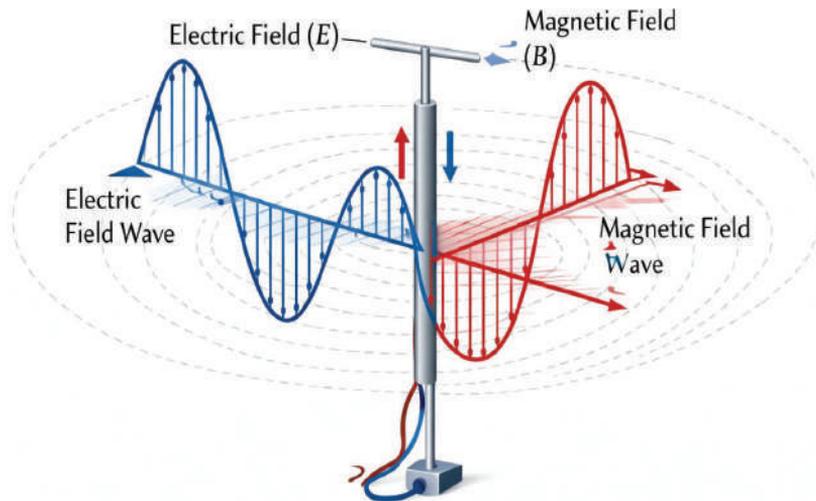
Figure 2.6: Propagation of electromagnetic waves through Antennas.

Later in 1908, Hermann Minkoski developed the tensor form of maxwell's equations, and Minkoski showed that, electric and magnetic field are not separated entities but components of a single electromagnetic filed tensor $F^{\mu\nu}$,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & -B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{2.11}$$

Where B= (B_x, B_y, B_z) is the magnetic field, E= (E_x, E_y, E_z) is electric field, and c represents speed of light. μ and ν represents Greek indices whose values is 0 for time and 1, 2, 3 for spatial components. Inhomogeneous equations (gauss's law for electricity + ampere law) reduce to one compact tensor form,

$$\partial_\nu F^{\mu\nu} = \mu_o J^\mu \tag{2.12}$$

Where ∂_ν represent the 4-gradient operator, $J^\mu = (c\rho, J)$ is the four-current (charge density ρ and current density J). Homogenous equations (Gauss's law for magnetism + Faraday law) in tensor form,

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{2.13}$$

Maxwell's unification of electricity and magnetism revealed that electric and magnetic fields are not separate phenomena but interconnected aspects of a single electromagnetic field. Through these equations, he showed how changing electric field generates magnetic field and changing magnetic field produces electric field leading to the prediction of electromagnetic waves. This unification explained the electromagnetic force as a fundamental interaction governing light, electricity and magnetism under one coherent framework.

Chapter # 03

General theory of relativity

In classical physics gravity is treated is a force that pulls objects toward each other. If the sun attract earth Newton says a force is acting between them. However, this explanation does not answer how the force travels through empty space or why it is must be instantaneous. Einstein proposed a deeper idea that gravity is not a force at all. Instead, gravity is the result of curvature of spacetime. Massive objects such as stars and planets bend spacetime around them, and other objects move along these bends. An object in free fall is not being pulled by gravity. It is simply following the straightest possible path in a curved spacetime. This is why astronauts in the orbit feel weightless they are freely fall in curved spacetime. This idea leads naturally to the mathematics of curved geometry. The geometry of spacetime is described by the metric tensor $g_{\mu\nu}$. The metric tells us how to measure the distance and time in curved geometry. The space time interval is written as Eq. (3.1),

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{3.1}$$

Where ds^2 represent the distance between two nearly events in spacetime, dx^μ represents small changes in space and time and $g_{\mu\nu}$ represents how much space is stretched or compressed and how time slowed or speed up. In plate space (no gravity) this equation reduces to the familiar formula of special relativity. In curved space it changes showing that gravity is present. So physically the metric is the gravitational field there is no separate gravitational force only geometry. In 1915 Einstein proposed his famous equation in general theory of relativity, Einstein field Eq. (3.2), replaced the Newtonian concept of gravity with geometric interpretation. Gravity arises from the curvature of spacetime caused by mass and energy is seen as in figure 3.1.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{3.2}$$

Where $G_{\mu\nu}$, is Einstein tensor, representing the spacetime curvature, it is constructed from Ricci curvature tensor and Ricci scalar tensor, and ensures local conservation of energy and momentum. Physically it describes how spacetime is bent or warped. $T_{\mu\nu}$ is the energy momentum tensor representing all form of energy and momentum, including mass density, pressure and energy flux. It acts as the source term of gravity. Λ is the cosmological constant it represents the energy density of empty space (vacuum energy). It plays a key role in modern cosmology particularly in explaining the accelerated expansion of the universe attributed to dark energy. The physical meaning of this equation is, matter and energy tell spacetime how to curve, and spacetime curvature tells matter how to move. Einstein field equation predict that time passes more slowly in stronger gravitational filed this phenomenon known as gravitational time dilation. Light follows curved paths in the presence of massive objects. This led to gravitational lensing where galaxies appear distorted. This phenomenon is a direct consequence of spacetime curvature described by Einstein field equation.
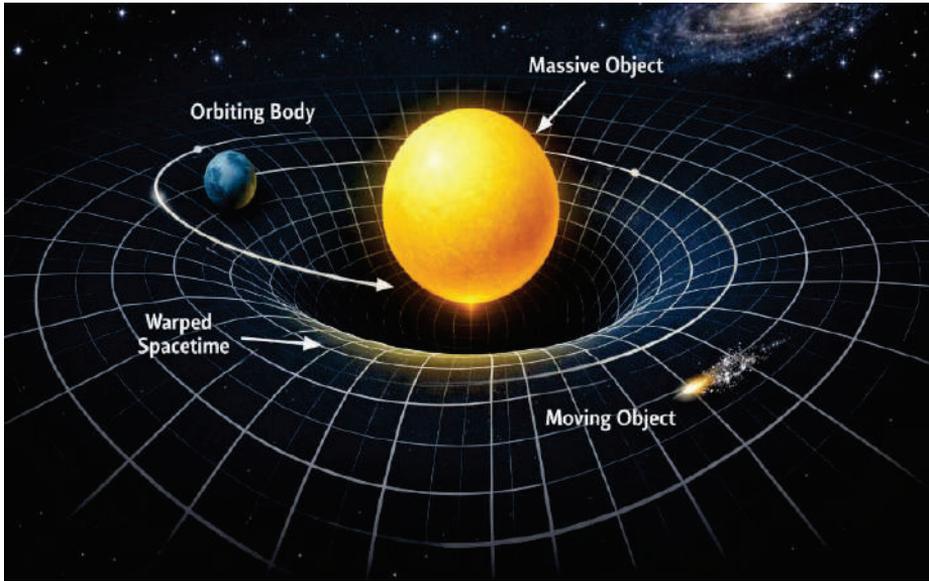
Figure 3.1: Representation of curved space time bent by mass to produce gravity.

In 1915 physicist Karl Schwarzschild solve the Einstein field equation for a static black hole is seen as in figure 3.2. When a very large mass is compressed into a very small region, spacetime becomes extremely curved, this leads to a black hole. Black holes represent one of the most fascinating and extreme prediction of general relativity, arising from gravitational collapse of massive stars or from elemental density fluctuation in the early universe. Define by a spacetime region from which nothing, not even light, can escape black holes are characterized by the presence of an event horizon enclosing a central singularity where classical description of physics breaks down. The fundamental properties of black holes are described by only a few parameters mass, angular momentum, and electric charge making them remarkably simple yet profound rich objects in theoretical and observational physics. The simplest black hole is described by Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{3.3}$$

The term $\left(1 - \frac{2GM}{c^2 r}\right)$, shows how time slows down and space stretches near massive object. As r becomes smaller times slow more and space becomes more distorted. When $r = \left(\frac{2GM}{c^2}\right)$, this term becomes zero. This radius is the event horizon: the boundary defining the region of space around a black hole from which nothing can escape. The Schwarzschild radius is $r_s = \left(\frac{2GM}{c^2}\right)$, is the point where escape velocity equals to the speed of light. Beyond this radius, light cannot escape, time effectively stops for an outer observer, all paths lead toward singularity: the point where the

curvature of spacetime is reached to infinity. Mathematically in the Schwarzschild metric for a non-rotating black hole, the component $\left(1 - \frac{2GM}{r}\right)$ is goes to infinity as r approaches zero, indicating infinite spacetime curvature. Where the laws of known physics break down.
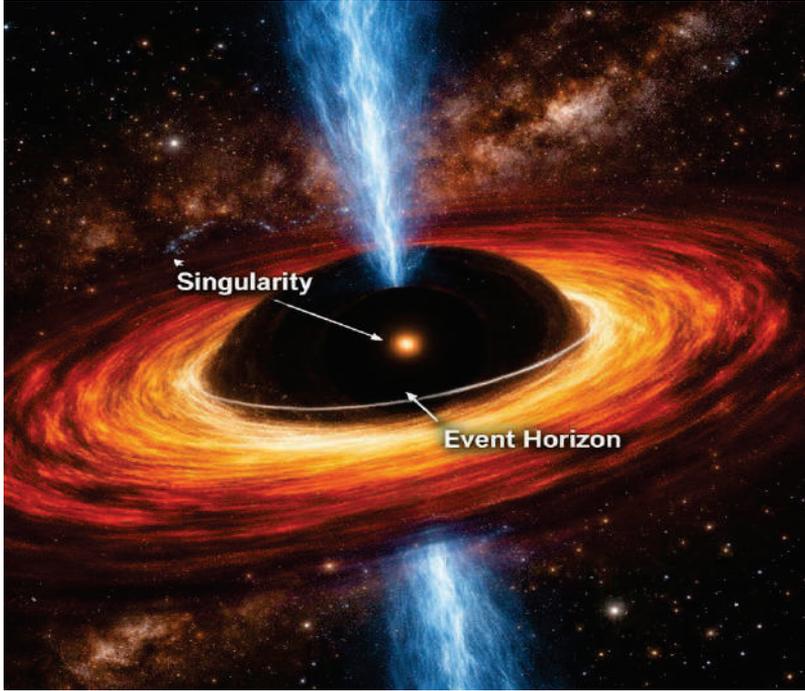
Figure 3.2 : Black hole: The point in the center represents singularity and the region outside represents the singularity.

White holes are described by time reversing $t \to -t$. The white hole as seen as in figure 3.3, is a theoretical solution spacetime solution of the field equations characterized by a region from which matter and radiations can only emerge, while no physical signal can enter. It arises as an exact vacuum solution of the field equations and is defined by a past singularity and an outgoing casual structure. The Einstein field equation given by Eq. (3.2), and for white hole spacetime the exterior region is assumed to be vacuum, such that $T_{\mu\nu} = 0$, and hence $G_{\mu\nu} = 0$. Solving these equations under the assumption of spherical symmetry and time independence leads to a unique spacetime geometry. The line element Eq. (3.4),

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2d(-t)^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\Omega^2 \qquad (3.4)$$

This metric represents the exact solution of field equations correspond to a white hole spacetime geometry. In spacetime the singularity is located at r =0, where curvature invariants diverge indicating a true physical singularity. The Kretchman scalar $K = 48G^2M^2/c^4r^6$, tends to infinity

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as r =0, conforming the presence of a past spacelike singularity from which spacetime emerges. The horizon is defined by the condition 1-2GM/c²r = 0, giving r = r$_s$. this surface represents a one-way casual boundary known as the white hole event horizon, across which matter and radiations can only move outward. The radial null geodesic given by Eq. (3.5),

$$\frac{dr}{dt} = \pm c \left(1 - \frac{2GM}{c^2 r}\right) \qquad (3.5)$$

Eq. (3.5), (+ for white hole while − for black hole). In white hole spacetime only outward solutions are physically allowed while inward trajectories are forbidden by the casual structure. As a result, all particles and light rays are repelled from the singularity and propagate outward through the horizon into the external universe.

Figure 3.3 : White hole: The central point represents the singularity and the region outside represent the event horizon.

Wormhole is a hypothetical tunnel like spacetime geometry connecting two distinct regions of the universe or two universes. In general relativity wormholes arise from the solution of the field equation under specific geometry and energy condition. A wormhole is described by Eq. (3.6),

$$ds^2 = -e^{2\Phi(r)}c^2 dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (3.6)$$

Where $b(r) = r_o^2/r$, where r$_o$ represents the throat radius. $\Phi(r)$, controls how time flows inside the wormhole, $b(r)$, controls the shape of the tunnel. When $b(r) < r$ the tunnel stays open and $b(r) > r$ the tunnel is closed. Physically this describes a bridge connecting two distinct regions of

space. To keep it open spacetime must be pushed outward this requires negative energy which is not known to exist naturally. In this geometry spacetime is smooth and finite everywhere there is no event horizon and time like and null geodesic can pass through the throat from one side to another. The curvature is remains finite at the throat, and the wormhole acts as a bridge rather a singular endpoint of spacetime, represent the structure of wormhole (Einstein-Rosen Bridge) is presented in figure 3.4.
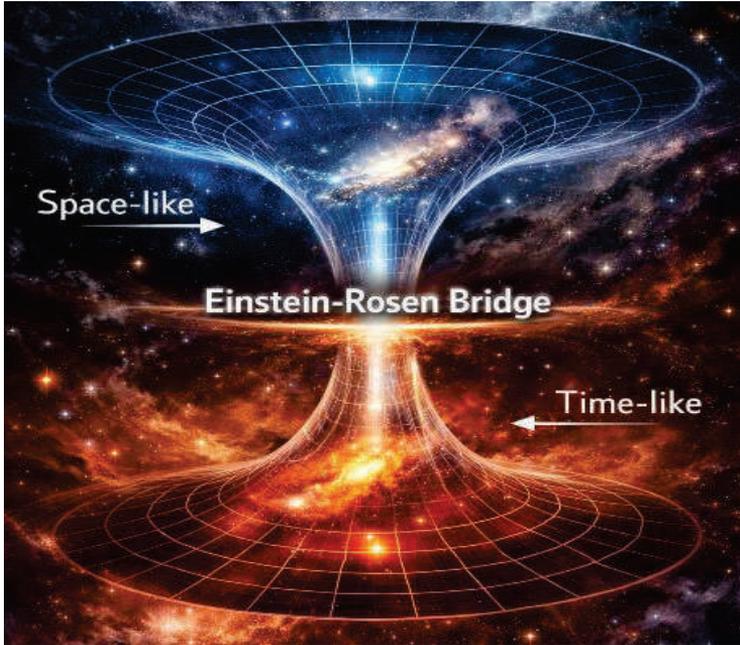
Figure 3.4: Representation of wormhole (Einstein-Rosen bridge).

The upper region in figure 3.4, represents how space is strongly curved near a massive object. Moving in this direction means moving through space at a fixed time. The space like region corresponds to spatial dimensions. Any slice taken here is a space like hypersurface. Where,

$$ds^2 > 0 \tag{3.6}$$

The lower region in figure 3.4, represents time like region represents temporal evolution. Movements in this direction means progression through time, not space. In strong gravitational field (such as near a black hole) time behaves differently and becomes a dominant coordinate. For time like paths,

$$ds^2 < 0 \tag{3.7}$$

The narrow central connection in figure 3.4, is the wormhole throat. It links two asymptotically flat spacetime regions. This bridge appears momentarily and collapses too fast to allow matter to

pass through (in classical general relativity). It is non-traversable without exotic matter. An Einstein-Rosen bridge is a mathematical connection between two spacetime regions where space-like and time-like coordinates change roles due to extreme curvature.
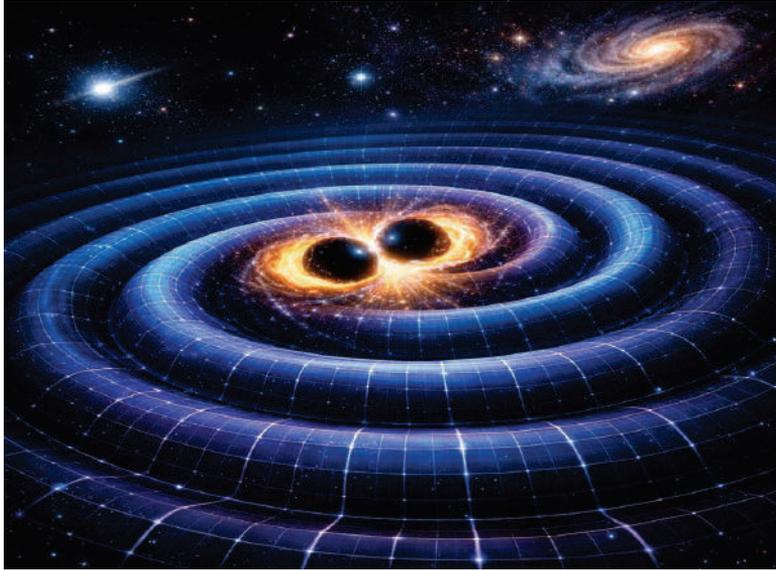
Figure 3.5: Propagation of gravitational waves arises from the collapsing of two black holes.

Gravitational waves are seen as in figure 3.5, arise as weak perturbation of spacetime that propagate according to field equation in vacuum. In the absence of energy and matter, the field equation reduces to Eq. (3.8),

$$G_{\mu v} = 0 \tag{3.8}$$

For a weak gravitation field, the space time metric can be expressed by Eq. (3.9),

$$g_{\mu v} = \eta_{\mu v} + h_{\mu v} \tag{3.9}$$

Where $\eta_{\mu v}$, is the Minkoski metric and $h_{\mu v} \ll 1$, represent a small perturbation. Linearized the field equations with respect to $h_{\mu v}$ and introducing the trace-reversed perturbation,

$$\bar{h}_{\mu v} = h_{\mu v} - \frac{1}{2}\eta_{\mu v}h \tag{3.10}$$

Where h_{α}^{α} the einstein field equations reduce to the homogenous wave Eq. (3.11),

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu v} = 0 \tag{3.11}$$

The solution of this equation has the plane wave solution given by Eq. (3.12),

$$\bar{h}_{\mu\nu} = A_{\mu\nu}e^{i(K_\alpha x^\alpha)} \tag{3.12}$$

Eq. (3.12) indicating that gravitational waves propagate at the speed of light. By choosing the transverse traceless gauge, the perturbation is constrained such that only two independent polarization modes, h_+ and $h_\times$ remain. For a wave propagating along the z-direction the metric perturbation takes the form,

$$h_{ij}^{TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos(wt - kz) \tag{3.13}$$

This equation leading to the line element,

$$ds^2 = -c^2dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + 2h_\times dxdy + dz^2 \tag{3.14}$$

Eq. (3.14), demonstrate that gravitational waves are transverse distortions of spacetime curvature that propagate as ripples through the vacuum, carrying energy and momentum away from accelerating mass distributions such as compact binary systems. General relativity successfully described the origin of gravity but it is always the classical theory of gravity.

Chapter # 04

Quantum Mechanics

Quantum mechanics is the fundamental theory that governs the behavior of matter and energy at atomic and subatomic scales. Classical physics which successfully explain macroscopic phenomena, fails completely when applied to microscopic systems. The stability of atoms, discrete spectral lines, electron diffraction, tunneling and the structure of matter cannot explain without quantum theory. The development of quantum theory in the early 20th century transformed our understanding of the nature. It revealed that matter and radiation do not behave according to common sense rules, but instead follow probabilistic laws governed by wavefunctions and operators. At the core of this theory lies the concept of wave particle duality, which states that every microscopic entity exhibits both wave and particle characteristics. The structure of atom, the behavior of electrons, and origin of chemical and physical properties are direct consequences of quantum mechanics. For centuries, matter was regarded as consisting of solid particles and light was understood as a wave. This clear distinction was destroyed by experimental discoveries. The photoelectric effect explain electrons are emitted from the surface of the metal when light of sufficient high frequency falls on it. When light strikes the metal, its energy is transformed to the electrons. If the energy of the electron is greater than the work function of the metal the electrons are ejected. Einstein explained this effect by proposing that light consist of small packets of energy called photons, given by Eq. (4.1),

$$E = hv \qquad (4.1)$$

Where E is the energy, h is plank's constant and v is the frequency. Photoelectric effect explains the particle nature of light and used in devices such as photoelectric cells, light sensor and solar panels. Electron diffraction experiment conform that matter has a wave nature. When a beam of electrons passed through a thin crystal or narrow slit it produces a diffraction pattern similar to light waves. So, electron diffraction confirms that matter behaves as a wave just like light. Louis de Broglie unified these ideas by proposing that every particle with momentum P, has a wavelength and given by Eq. (4.2),

$$\lambda = \frac{h}{p} \qquad (4.2)$$

This hypothesis introduced the idea of matter waves. According to this view, electrons, protons, and even atoms possess wave properties, through their wavelength are usually extremely small is seen as in figure 4.1.
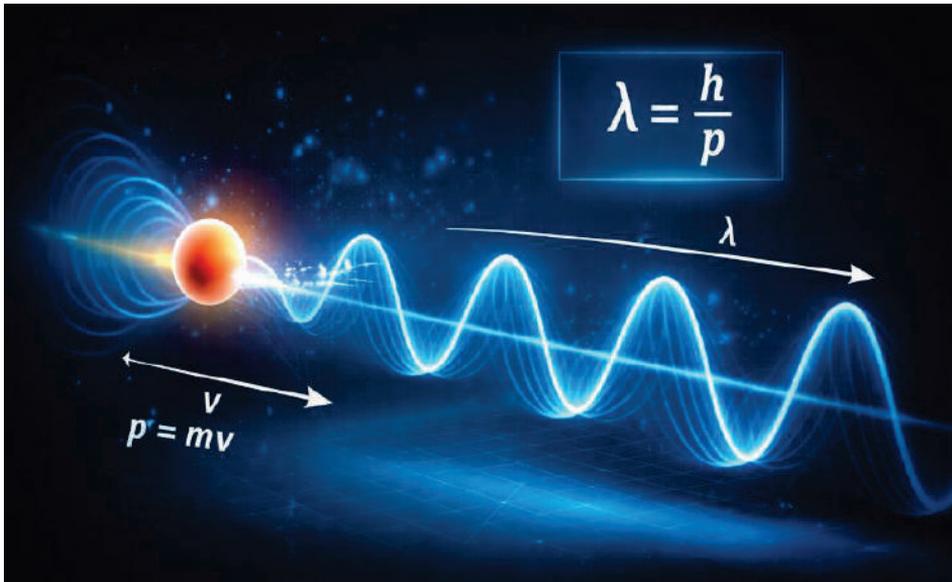
Figure 4.1: Representation of matter waves.

The double slit experiment with electrons is a core stone of quantum mechanics and provides direct experimental evidence for dual nature of matter, demonstrating that electrons exhibits both wave-like and particle-like properties. Figure 4.2, displayed the double-slits experiment where a beam of electrons is emitted from an electron gun and directed towards a barrier containing two closely spaced narrow slits. Beyond the slits a fluorescent screen is placed to record the arrival of electrons. When electrons strike the screen, each electron is detected as a localized point conforming the particles nature of the electron. However, when a large number of electrons are allowed to pass through the two slits, interference pattern consisting of alternating bright and dark fringes gradually emerges on the screen. This pattern is characteristic of wave interference and cannot be explained by classical particle theory. Furthermore, if a measuring device is placed at the slits to determine through which slit the electron passes, the interference pattern disappears and only two bright bands corresponding to the two slits are observed. This demonstrates that the act of observation forces the electron to behaves as a particle destroying its wave nature. Therefore, the double slit experiment proves the principles of wave particle duality establishing that electrons, like light, possess both wave and particle characteristics. This experiment proves the de Broglie's hypothesis of matter waves.
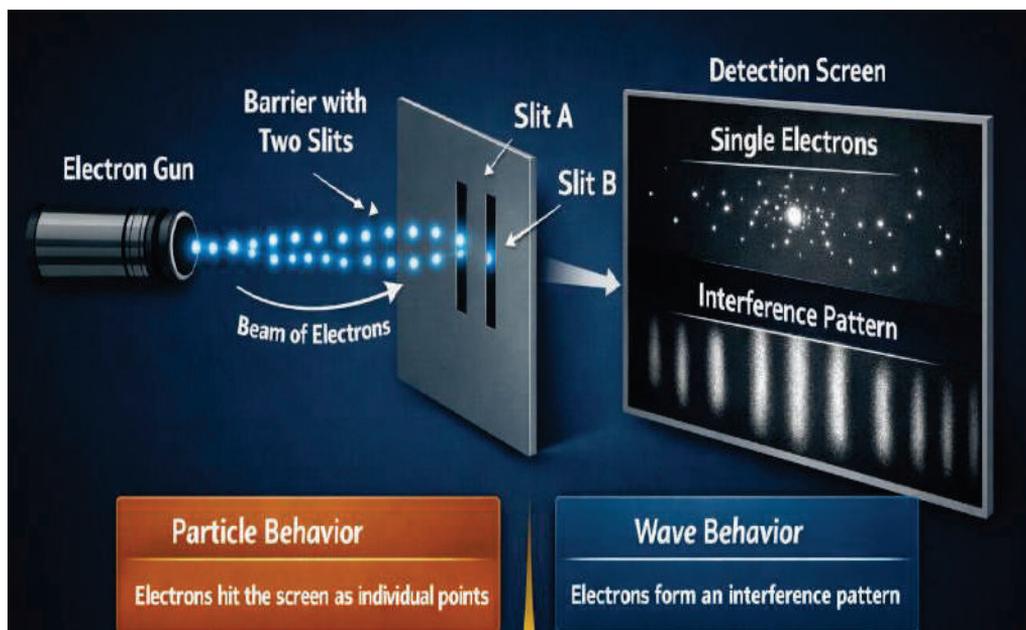
Figure 4.2: Experimental setup of Double slits experiment, where electrons show dual behavior.

The Heisenberg uncertainty principle, proposed by Werner Heisenberg in 1927, is a fundamental principle in quantum mechanics which states that it is impossible to simultaneously determine the exact position and exact momentum of a particle with absolute certainty. The more precisely the position of a particle is known, the less precisely its momentum can be known and vice versa. This is not due to experimental limitations but is an inherent property of nature at the quantum scale. Mathematically can be written as,

$$\Delta x \, \Delta p \geq \frac{h}{4\pi} \tag{4.3}$$

Where the Δx, is the uncertainty in position and Δp is the uncertainty in momentum. This principle implies that electron do not move in well-defined orbits around the nucleus as suggested by earlier atomic models but rather exist in regions in probability.

Dalton atomic model is seen in figure 4.3 (a), described the atom as a solid, invisible sphere and laid the foundation of modern atomic theory. Later J.J Thomson discovered the electron and proposed the plum pudding model is seen as in figure 4.3 (b), in which negatively charged electrons were embedded in positively charged sphere. Rutherford model is seen as in figure 4.3 (c), through his gold foil experiment, revealed that the atom has a small, dense, positively charged nucleus with electrons revolving around it indicating that most of the atom is empty space. Bohr improved this model is seen as in figure 4.3 (d), by introducing fixed quantized energy levels in which electrons move without radiating energy successfully explaining the hydrogen spectrum. Sommerfeld model

is seen as in figure 4.3 (e), further refined Bohr's model by allowing electrons to move in elliptical orbits and introducing sun-energy levels. Schrodinger developed the quantum mechanical model is seen as in figure 4.3 (f), where electrons are described by wave function and exist in probability regions called orbitals rather than fixed paths, which is the most accurate and accepted model today. Finally, Chadwick model is seen as in figure 4.3 (g), provide the evidence that the neutron in nucleus, completing the picture of atomic structure and explaining isotopes. Thus, the modern atom is understood as a dense nucleus of proton and neutrons surrounded by electrons distributed in orbitals governed by quantum mechanics.
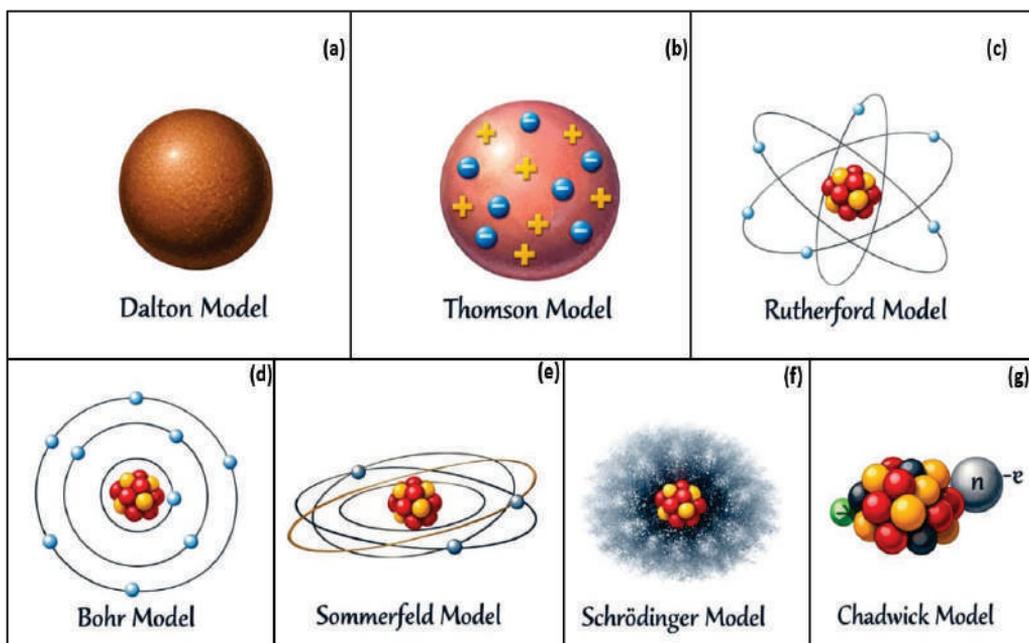
Figure 4.3: Representation of atomic models (a) Dalton model (b) Thomson model (c) Rutherford model (d) Bohr model (e) Sommerfeld model (f) Quantum mechanical model (g) Chadwick model respectively.

The wavefunction denote by Ψ is the fundamental quantity that completely describe the state of a particle or system. Unlike classical physics where position and velocity well defined, quantum mechanics uses the wavefunction to know all possible information about a particle such as its position, momentum and energy. The wavefunction can be complex and its behavior governed by the Schrodinger equation. The general representation of wavefunction given by Eq. (4.4),

$$\Psi(x, t) = A e^{i(kx - \omega t)} \tag{4.4}$$

Where A represent the amplitude, k represents the wave number and ω represents the angular frequency. The square of the wavefunction is probability density and given by Eq. (4.5),

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) \tag{4.5}$$

Where Ψ^* is the complex conjugate of Ψ. According to Max Born's interpretation, $|\Psi(x,t)|^2 dx$ gives the probability of finding a particle between position x and x+dx at time t. this means quantum mechanics is inherently probabilistic, not deterministic like classical physics. The total probability must be equal to 1.

$$\int_{-\infty}^{+\infty}|\Psi(x,t)|^2 dx = 1 \tag{4.6}$$

This is called normalization of the wavefunction. The fundamental equation of quantum mechanics is the Schrodinger equation it describes how the wavefunction evolves in time.

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \tag{4.7}$$

Where H represents the Hamiltonian of the system. Hamiltonian is the total energy of the system.

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r,t) \tag{4.8}$$

This equation plays the same role in quantum mechanics as Newton's second law in classical physics. In quantum mechanics a system can exist in multiple states simultaneously. This principle is known as superposition states. If $|\psi_1\rangle$ and $|\psi_2\rangle$ are the two possible states of a system, then most general state is a linear combination of these states.

$$|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \tag{4.9}$$

Where c_1 and c_2 are complex probability amplitudes, $|c_1|^2$ and $|c_2|^2$ gives the probabilities of finding the system in $|\psi_1\rangle$ or $|\psi_2\rangle$. For example, for spin (-1/2) particles,

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \tag{4.10}$$

This means that particle in both spin-up and spin-down state at the same time until a measurement is made. Upon measurement the wavefunction collapses into one of the eigenstates. Quantum entanglement is seen as in figure 4.5, is a phenomenon in which two or more particles become so strongly correlated that the state of one particle effects the state of another particle no matter how far apart they. A famous example is the Bell state,

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.11}$$

Where $|00\rangle$ represent both particles in state 0, and $|11\rangle$ represents both particles in state 1, so the system in both states simultaneously. Another example is,

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{4.12}$$

Here if A particle is measured and found in state $|0\rangle$, particle B immediately be in state $|1\rangle$, even if they are separated by large distances.
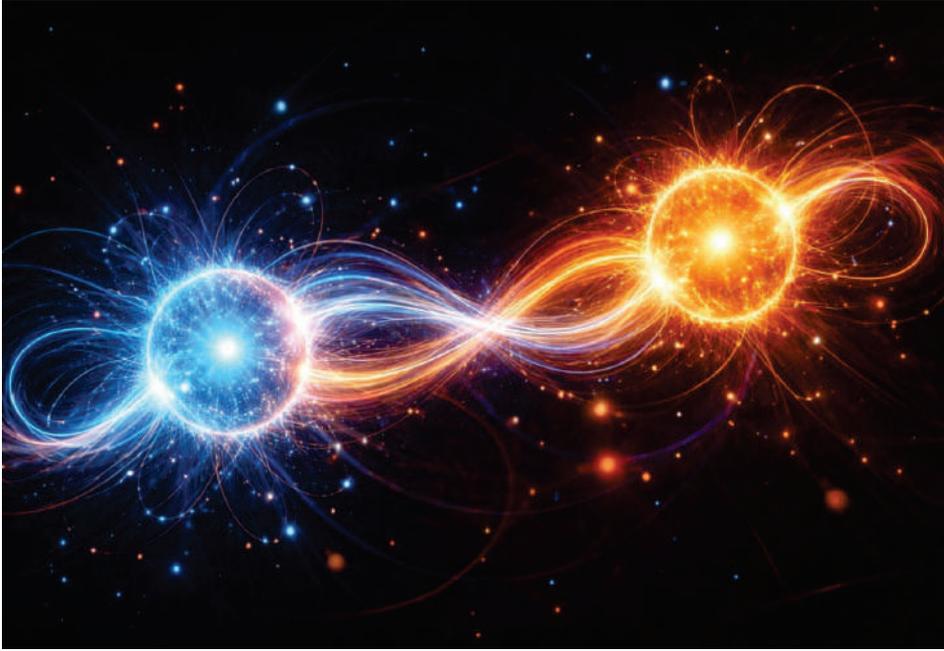
Figure 4.5: Quantum entanglement where two photons are entangled at larger distance.

Quantum teleportation is displayed in figure 4.6, is the process of transferring an unknow quantum state from one location to another location using quantum entanglement without physically sending the particle. Alice has a qubit in unknow state,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{4.13}$$

Alice and bob share an entangled pair,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.14}$$

Combined initial state,

$$|\Psi\rangle total = |\psi\rangle \otimes |\Phi^+\rangle \tag{4.15}$$

$$|\Psi\rangle total = \alpha|0\rangle + \beta|1\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.16}$$

After Alice performs a bell state measurement bob's qubit becomes,

$$|\psi\rangle B = \sigma_i(\alpha|0\rangle + \beta|1\rangle) \tag{4.17}$$

Where $\sigma_i \in (I, X, Y, Z, XZ)$ are pauli operators depending on Alice measurement result. Bob applies the correct unitary operation to recover the original state,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{4.18}$$
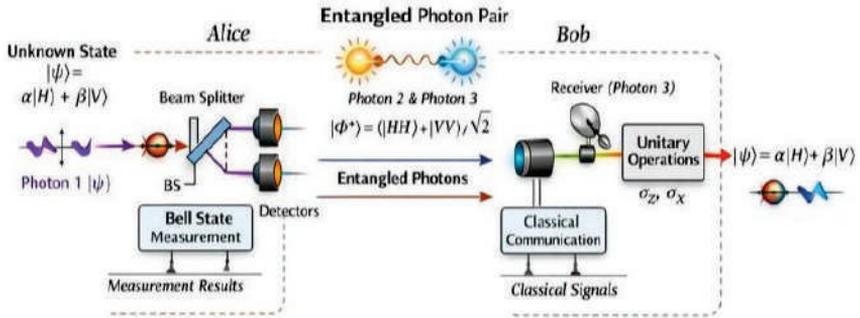
Figure 4.6: Experimental setup of Quantum teleportation by entangles photons between Alice and Bob.

Chapter # 05

Relativistic quantum mechanics

Relativistic quantum mechanics (RQM) is the theoretical framework that unifies the principle of quantum mechanics with special theory of relativity. While non relativistic quantum mechanics (based on the Schrodinger equation) successfully describes microscopic systems at low velocities, it fails when particles move near the speed of light ($v \cong c$). In such regimes, relativistic effects such as time dilation, length contraction, and the equivalence of mass and energy becomes important. RQM, provides the correct description of high energy particles, antiparticles, spin and phenomena such as particle creation and annihilation. The development of RQM, was historically driven by the need to reconcile quantum theory with Einstein special theory of relativity. In special relativity the total energy E, and momentum p, of a particle with the rest mass m, are related by Eq. (5.1),

$$E = p^2c^2 + m^2c^4 \tag{5.1}$$

This equation is replaced classical kinetic energy and central to all relativistic equations. The time dependent Schrodinger equation is,

$$i\hbar \frac{\partial \psi}{\partial t} = (-\frac{\hbar^2}{2m}\nabla^2 + V)\psi \tag{5.2}$$

Eq. (5.2), is not Lorentz invariant and assumes that kinetic energy is $p^2/2m$, which is only valid for velocities much smaller than c. Therefore, a new formulation requires for relativistic particles. Starting from Eq. (5.1) and using the quantum mechanical operators,

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad p \rightarrow i\hbar\nabla \tag{5.3}$$

We obtain,

$$\left(i\hbar \frac{\partial}{\partial t}\right)^2 \phi = (-\hbar^2c^2\nabla^2 + m^2c^4)\phi \tag{5.4}$$

Which leads to the Klein-Gorden equation for relativistic particles,

$$(\partial_\mu \partial^\mu + m^2)\phi = 0 \tag{5.5}$$

Where $\partial_\mu \partial^\mu$ represents the four derivatives of spacetime, m is the mass of the particle and ϕ represents the scalar field. This equation is invariant under Lorentz transformations and thus fully consistent with the principles of special relativity. The Klein Gorden equation describes the scalar particles (particles with zero spin i.e. pions and Higgs boson). The solution of the Klen Gorden equation given by Eq. (5.6),

$$\psi(r,t) = Ae^{i(\vec{k}.\vec{r}-\omega t)} \tag{5.6}$$

Where A represents the amplitude, k is the wavenumber, t is the time and ω represent the angular frequency. When substitute Eq. (5.6) into Eq. (5.5) it gives the relation,

$$\omega^2 = c^2 k^2 + \frac{m^2 c^4}{\hbar^2} \tag{5.7}$$

Eq. (5.7) equivalent to the relativistic energy momentum relation. This confirms that the equation correctly represents relativistic dynamics at the quantum level. However, the Klein Gorden equation exhibits both positive and negative energy solutions,

$$E = \pm\sqrt{p^2 c^2 + m^2 c^4} \tag{5.8}$$

Initially the presence of negative energy states posed serious conceptual difficulties, as they seemed to imply the possibility of particles possessing arbitrarily low energies. This problem was later resolved within the framework of quantum field theory where negative energy solution is reinterpreted as corresponding to antiparticles. A major limitation of the Klein Gorden equation lies in its probability interpretation. In nonrelativistic quantum mechanics the probability density is $|\psi|^2$ which is always positive. In contrast, the conserved density associated with Klein Gorden is given by Eq. (5.9),

$$\rho = \frac{i\hbar}{2mc^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \tag{5.9}$$

According to Eq. (5.9), the density is not guaranteed to be positive. This mean that it cannot be consistently interpreted as a probability density, decline the physical meaning of ψ, as a probability amplitude. Another limitation of Klein Gorden equation it is only work for spin zero article and fail to explain the fermions like electron, which is spin (1/2) particles. As a result, it cannot account for essential physical phenomena such as the electron magnetic moment, spin-orbit coupling and fine structure of atomic spectra. In 1928 Paul Dirac give a first order relativistic wave equation, which is liner in time and space derivative, is consistent with special relativity, naturally include electron spin, and gives positive define probability density. Dirac proposed that the relativistic energy relation should de linearized as,

$$E = c\alpha.P + \beta mc^2 \tag{5.10}$$

Replacing energy and momentum by operators,

$$i\hbar \frac{\partial \psi}{\partial t} = (-\hbar c\alpha.\nabla + \beta mc^2)\psi \tag{5.11}$$

This is the Dirac equation. Where ψ is a four-component spinor wavefunction, $\alpha = (\alpha_x, \alpha_y, \alpha_z)$ and β are 4×4 are matrices, m is the rest mass and c are the speed of light. In standard representation,

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{5.12}$$

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{5.13}$$

Where σ_i are pauli matrices and I is 2×2 matrix. The wavefunction ψ has four components,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{5.14}$$

Two components represent spin-up and spin-down electron while two components represent negative energy solution, later interpreted as positron (antiparticles), thus the Dirac equation predicts anti matter a major success of the theory. The probability density is,

$$\rho = \psi^* \psi \tag{5.15}$$

And the probability current is,

$$J = c\psi^\dagger \alpha \psi \tag{5.16}$$

The satisfy the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \tag{5.17}$$

This gives the conservation of probability which solves the problem of Klein-Gorden equation. The final form of the Dirac equation,

$$\left(i\gamma^\mu \partial_\mu - m \right)\psi = 0 \tag{5.18}$$

Where γ^μ are gamma matrices, with repeated values, $\mu = 0,1,2,3$ and ∂_μ is the four gradients. The solution of the Dirac equation for free particle is,

$$\psi = u(p)e^{-\frac{i}{\hbar}(P.r - Et)} \tag{5.19}$$

Where $u(p)$ represent the four-components spinor, the substitution of Eq. (5.19) into Dirac equation gives,

$$E = \pm\sqrt{p^2 c^2 - m^2 c^4} \tag{5.20}$$

The positive solution corresponds to electron while the negative solution corresponds to positrons. The hydrogen atom spectrum shows that each energy level predicted by Schrodinger equation is actually split into several closely spaced lines known as fine structure. This splitting arises dua to relativistic effects that are not accounted for in non-relativistic quantum mechanics. The origin of fine structure can be traced to three physical effects: relativistic correction to kinetic energy of the electron, the interaction between electron's spin and its orbital motion known as spin-orbit coupling, and the Darwin term which arises from rapid quantum fluctuation in the electron's position. In the Schrodinger framework these effects are introduced artificially as perturbations. In

contrast Dirac equation incorporates all of them naturally and providing a complete unified explanation. For an electron moving in the Coulomb field of the nucleus, the potential energy is given by,

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_o r} \tag{5.21}$$

The Dirac equation in the central field potential is,

$$c\alpha.P + \beta mc^2 - \frac{Ze^2}{4\pi\varepsilon_o r} \tag{5.22}$$

Eq. (5.22) is the relativistic wave equation for hydrogen atom. Solving this equation leads to the energy eigen values,

$$E_{n,j} = mc^2 \left[1 + \frac{(Z\alpha)^2}{\left(n-j-\frac{1}{2}+\sqrt{\left(j+\frac{1}{2}\right)^2-(Z\alpha)^2}\right)^2}\right]^{-1/2} \tag{5.23}$$

Where n is the principle quantum number, j is the total angular momentum quantum number, and α is the fine structure constant. This single expression correctly predicts the splitting of spectral lines and depends explicitly on the total angular momentum j, showing that energy levels are no longer degenerate in l as in the Schrodinger theory. The first contribution to fine structure is the relativistic correction to kinetic energy. In classical mechanics the kinetic energy is $P^2/2m$ but in relativity it is given by Eq. (5.24),

$$E = \pm\sqrt{p^2c^2 - m^2c^4} \tag{5.24}$$

When expand for small p this becomes,

$$E = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} \tag{5.25}$$

The term in Eq. (5.25) represents the relativistic correction. The second contribution is the spin orbit coupling. From a physical point of view in the rest frame of electron the nucleus appears to be moving and thus produces a magnetic field. The magnetic movement associated with the electron's spin interacts with this magnetic field leading to an additional interaction energy. The spin orbit Hamiltonian is given by Eq. (5.26)

$$H_{SO} = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}L.S \tag{5.26}$$

For the coulomb potential this becomes,

$$H_{SO} = \frac{Ze^2}{8\pi\varepsilon_o m^2c^2r^3}L.S \tag{5.27}$$

This term causes the energy levels to split depending on whether the spin and orbital angular momentum are aligned or anti-aligned. The third contribution is the Darwin term which has no classical analogue. It arises due to the rapid vibrating motion of the electron known as zitterbewegung predicted by the Dirac equation. The Darwin Hamiltonian is,

$$H_D = \frac{\hbar^2}{8m^2c^2} \nabla^2 V \tag{5.28}$$

For the coulomb potential this becomes a delta function term that effects only s-state where the probability density at the nucleus is non zero. In nonrelativistic quantum mechanics the total fine structure correction is,

$$\Delta E_{FS} = \langle H_{rel} \rangle + \langle H_{SO} \rangle + \langle H_D \rangle \tag{5.29}$$

Which leads to,

$$\Delta E_{FS} = \frac{E_n^2}{2mc^2} \left[\frac{3}{4n} - \frac{1}{j+\frac{1}{2}} \right] \tag{5.30}$$

Where E_n is the Bohr's energy. This show explicitly that the energy depends on the total angular momentum quantum number j, explaining the observed splitting. As a result, the Dirac equation provides an exact explanation of the fine structure of the hydrogen atom and predicts the correct g-factor of electron. This unification of relativity, spin, and quantum mechanics stands as one of the greatest attempts of the theoretical physics and definitely established the Dirac equation as the correct relativistic description of spin (1/2) particles.

Chapter # 06

Quantum field theory

Quantum field theory (QFT), is the theoretical framework which unifies quantum theory, special relativity and field theory and provides the most accurate description of nature at the microscopic level. In quantum field theory the basic idea is fields nor particles. A field is physical quantity that has a value (scalar, vector or tensor) at every point in space and time representing how a force or physical property is distributed. Particles are interpreted as quantized excitations of these fields. This view point naturally explains particle creation and annihilation, relativistic invariance, and interactions through force carrying fields. In nonrelativistic quantum mechanics a particle is described by a wavefunction $\psi(x,t)$ whereas, in QFT, this wavefunction is replaced by field operators such as $\hat{\phi}(x)$, $\hat{\psi}(x)$, and $\hat{A}_\mu(x)$. Every type of particles corresponds to a particular field, and depending on how these fields transform under Lorentz transformations. A scalar field is seen as in figure 6.1 (a), assigns a single numerical value to every point in spacetime and remains unchanged under Lorentz transformation: $\phi'(x') = \phi(x)$. Scalar fields describe spin-0 particles such as Higgs boson. The dynamics of the scalar field are governed by the Klein-Gorden lagrangian,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 \tag{6.1}$$

Applying Euler-Lagrangian Eq. (6.2), to the Klein-Gorden lagrangian,

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0 \tag{6.2}$$

Leads to Klein-Gorden equation,

$$(\partial_\mu\partial^\mu + m^2) = 0 \tag{6.3}$$

To quantize the scalar field, one introduces the canonical conjugate momentum associated with the field, defined as,

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_o\phi)} = \dot{\phi}(x) \tag{6.4}$$

Using the filed ϕ, and its conjugate momentum π, the Hamiltonian density is constructed as,

$$\mathcal{H} = \dot{\phi}\pi - \mathcal{L} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 \tag{6.5}$$

The total Hamiltonian is obtained by integration density in Eq. (6.5), over all space and represent the total energy of the scalar field. Canonical quantization is achieved by promoting the classical field variable $\phi(x)$ and $\pi(x)$, to operators acting on a Hilbert space. These operators satisfy equal-time commutation relations,

$$\left[\hat{\phi}(t,x),\hat{\pi}(t,y)\right] = i\delta^{(3)}(x-y) \tag{6.6}$$

While the commutators of the field with itself and the momentum with itself vanish. These relations are the field-theoretic generalization of the canonical commutation relations of quantum mechanics. The field operator is then expanded in terms of normal modes. For a free scalar field this expansion takes the form,

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_P}} \left(\hat{a}_p e^{-ip.x} + \hat{a}_p^\dagger e^{ip.x}\right) \tag{6.7}$$

Where $E_P = \sqrt{p^2 + m^2}$. The operators $\hat{a}_p$ and $\hat{a}_P^\dagger$ are identified as annihilation and creation operators respectively, and they satisfy the commutation relations,

$$[\hat{a}_p, \hat{a}_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(P-P') \tag{6.8}$$

With all other commutators vanishing. These relations ensure consistency with the canonical commutation rules of the field operators. When the mode expansion is substituted into the Hamiltonian the Hamiltonian operator becomes,

$$\hat{H} = \int \frac{d^3P}{(2\pi)^3} E_P \left(\hat{a}_P^\dagger \hat{a}_p + \frac{1}{2}\right) \tag{6.9}$$

This expression shows that each momentum mode of the scalar field behaves like an independent quantum harmonic oscillator. The term proportional to $\frac{1}{2}E_P$, represents the zero-point energy of the vacuum. The physical interpretation of the quantized scalar field emerges naturally from this formalism. The vacuum state $|0\rangle$ is defined as the state annihilated by all annihilation operators, $\hat{a}_p|0\rangle = 0$. Acting with a creation operator on the vacuum produces a one-particle state of define momentum, $|P\rangle = \hat{a}_P^\dagger|0\rangle$. Thus, particles are interpreted as quanta of the underlying scalar field, and multi-particles states arise from repeated applications of creation operators.
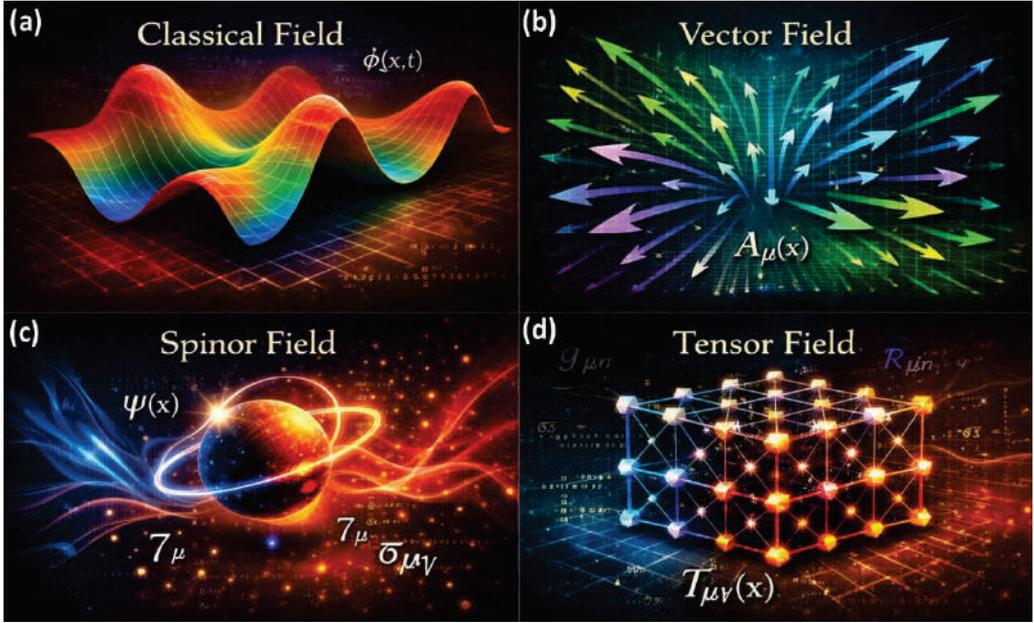
Figure 6.1: Representation of (a) Classical field (b) Vector field (c) Spinor field and (d) Tensor field.

A vector field assigns a four-vector to each point in spacetime and transform as,

$$A'_\mu = \Lambda_\mu^v A_v(x) \tag{6.10}$$

Vector fields describe spin-1 particles and are responsible for mediating forces. The electromagnetic field is seen in figure 6.1 (b), is the most familiar example, represented by the four-potential A_μ. The corresponding field strength tensor is defined as,

$$F_{\mu v} = \partial_\mu A_v - \partial_v A_\mu \tag{6.11}$$

The lagrangian density for a free vector field is define as,

$$\mathcal{L} = -\frac{1}{4} F_{\mu v} F^{\mu v} \tag{6.12}$$

When substitute the lagrangian density from Eq. (6.12), in the Euler-Lagrangian Eq. (6.2), yields the maxwell equation,

$$\partial_\mu F_{\mu v} = 0 \tag{6.13}$$

Quantization of the vector field requires imposing commutation relations between the field and its conjugate momentum, leading to the field expansion,

$$\hat{A}_\mu(x) = \sum_{\lambda=1}^{2} \int \frac{d^3P}{(2\pi)^3 2E_P} \left[\varepsilon_\mu^{(\lambda)} \hat{a}_P^{(\lambda)} e^{-ipx} + \varepsilon_\mu^{(\lambda)*} \hat{a}_P^{(\lambda)\dagger} e^{ipx} \right] \tag{6.14}$$

Here, the polarization vectors $\varepsilon_\mu^{(\lambda)}$, describe the two physical transverse photon states, and the operator $a^\dagger$ and a create and annihilate photons, showing that photons are quanta of the electromagnetic field.

Spinor field is seen in figure 6.1 (c), describe particles with half -integer spin such as electrons, quarks, and neutrinos. A spinor field is represented by a four-component object $\psi(x)$, that transform according to the spinor representation of the Lorentz group,

$$\psi'(x') = S(\Lambda)\psi(x) \tag{6.15}$$

The dynamics of a free spinor field are given by the Dirac-lagrangian density,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{6.16}$$

Where $\bar{\psi} = \psi^\dagger \gamma^o$, applying the Euler-Lagrange equation to the Dirac- lagrangian density leads to the Dirac equation for spin (1/2) particles.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{6.17}$$

The gamma matrices satisfy the anticommutation relation Eq. (6.18), ensuring Lorentz invariance.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I \tag{6.18}$$

Quantization of the spinor field requires anticommutation relations because fermions obey the pauli exclusion principle. The canonical anticommutation relations is,

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \delta_{\alpha\beta}\delta^3(x-y) \tag{6.19}$$

The field operator is expended as,

$$\hat{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \left[b_P^s u^s(p) e^{-ipx} + d_P^{s\dagger} v^s(p) e^{ipx} \right] \tag{6.20}$$

Where $b^\dagger$ creates fermions and $d^\dagger$ creates antifermions. This show that matter particles and their antiparticles arise as excitation of spinor field.

Tensor field is seen as in figure 6.1 (d), assign tensors to each point in spacetime and describe higher-spinor fields and spacetime structure. The most important example is the gravitational field, represented by the metric tensor $g_{\mu\nu}(x)$. In the weak field limit, the metric can be written as,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6.21}$$

Where $h_{\mu\nu}$ is a small perturbation around flat spacetime. This perturbation behaves like a spin-2 field and associated with the gravitation. The lagrangian density for linearized gravity can be written as,

$$\mathcal{L} = \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial^\lambda h_{\lambda\nu} \tag{6.22}$$

Quantization of the tensor field leads to the expansion,

$$\hat{h}_{\mu\nu}(x) = \sum_s \int \frac{d^3P}{(2\pi)^3 2E_P}[\varepsilon^s_{\mu\nu}a^s_P e^{-ipx} + \varepsilon^{s*}_{\mu\nu}a^{s\dagger}_P e^{ipx}] \tag{6.23}$$

Where $a^\dagger$ and a create and annihilate gravitons. Although a complete quantum theory of gravity is still under development this approach shows how gravity can be treated as a quantum field in the weak field limit.

Quantum electrodynamics (QED), unifies the electric and magnetic forces at the quantum level through fundamental interaction. Gauge theory states that fundamental interaction arises from requiring invariance under symmetry transformations. Consider the free Dirac field describing charged fermions,

$$\mathcal{L}_o = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{6.24}$$

The lagrangian is invariant under a global phase transformation,

$$\psi(x) = e^{i\alpha}\psi(x) \tag{6.25}$$

Promoting the symmetry to a local transformation breaks invariance because the derivative changes as,

$$\partial_\mu\psi \to (\partial_\mu + i\partial_\mu\alpha)\psi \tag{6.26}$$

To restore invariance a gauge field A_μ is introduced and the derivative is replaced by the covariant derivative,

$$D_\mu = \partial_\mu + ieA_\mu \tag{6.27}$$

The gauge field transform as,

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{6.28}$$

Ensuring the gauge invariance of the theory. The electromagnetic field dynamics are governed by the gauge-invariant filed tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.29}$$

With kinetic term the lagrangian density becomes,

$$\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.30}$$

Combining gauge filed and matter field yield the lagrangian of quantum electrodynamics,

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.31}$$

This single equation describes charged particles, electric interaction, magnetic interaction and electromagnetic radiations. Thus, electric and magnetic forces are unified within electromagnetic force. The interaction term,

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \tag{6.32}$$

Defines the electromagnetic current,

$$J^{\mu} = e\bar{\psi}\gamma^{\mu}\psi \tag{6.33}$$

In QED, forces arise through photon exchange between charged particles is represented by Feynman's diagram. A Feynman's diagram introduced by Richard Feynman, is a graphical representation used in quantum field theory to describe how particles interact. In the case of electromagnetic interaction, it explains how charged particles interact through the exchange of photon. For example, in figure 6.2, when electrons interact, they do not directly touch each other, instead they exchange a virtual photon, which acts as the mediator of the electromagnetic force. In the diagram the straight lines represent the electrons (fermions) while a wavy line represents the exchange photon. The point where these lines meet is called a vertex and indicates the interaction between the charged particle and the photon. Although the diagram looks like a simple drawing, it mathematically represents the probability amplitudes of the interaction process rather than the exact physical path of the particles. The coulomb potential emerges naturally from one photon exchange,

$$V(r) = \frac{e^2}{4\pi r} \tag{6.34}$$

Magnetic interaction appears as relativistic corrections,

$$F_{mag} \sim \frac{v^2}{c^2} F_{electric} \tag{6.35}$$

Hence electric and magnetic forces are not independent they are frame-dependent manifestation of the same quantum interaction.
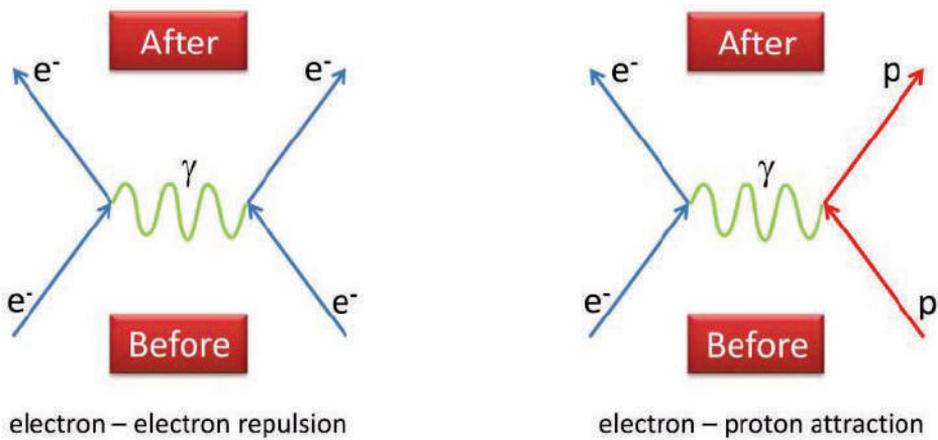
Figure 6.2: Representation of interaction of electromagnetic charges through photon exchange by Feynman's diagrams.

The electroweak theory was developed to explain the deep connection between electromagnetic and weak interaction which experimentally appear very different at low energies but reveal a common origin at high energies. Electromagnetic force is a long-range force mediated by a massless photon and conserved parity, while the weak interaction is short-range, mediated by massive vector bosons (W and Z) represented by Feynman's diagrams is seen as in figure 6.3. A Feynman diagram for weak interaction represents the process in which particles interact through the exchange of heavy gauge bosons. In the diagram fermions such as quarks or leptons are shown as straight lines, while the weak force is mediated by a wavy line corresponding to the W and Z boson. For example, in beta decay a down quark changes into up quark by emitting a W⁻ boson, which then decays into an electron and antineutrino. This illustration that weak interaction can exchange the flavor of particles and is responsible for processes like radioactive decay. Despite these differences, both interactions show similar coupling strengths at high energies and share a common current structure. This motivated idea that they arise from a single gauge symmetry that is spontaneously broken in nature. The electroweak interaction described by a non-Abelian gauge theory based on the symmetry group $SU(2)_L \times U(1)_Y$, where $SU(2)_L$ represent weak isospin acting only on left-handed fermions and $U(1)_Y$ represents weak hypercharge acting on both left- and right-handed fermions.
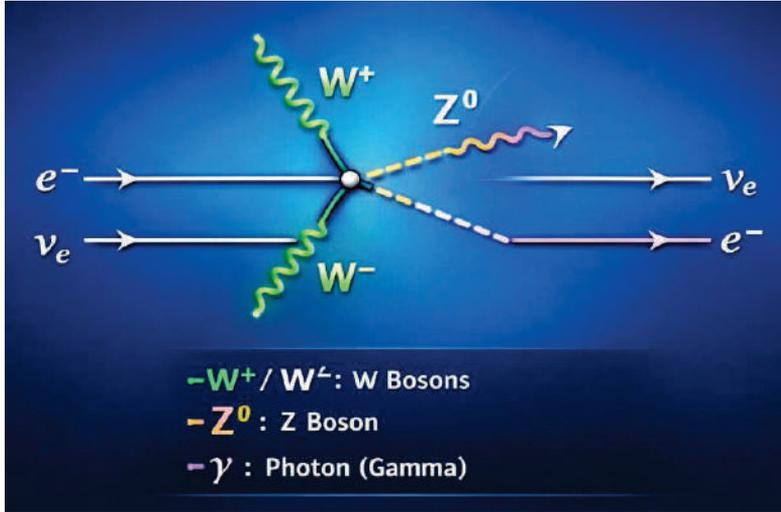
Figure 6.3: Representation of Feynman diagram for electroweak interaction mediated by W and Z bosons.

Fermions are arranged according to their transformation properties under this gauge group. Left-handed fermions appear in weak isospin doublets, such as $L = (v_e, e)_L^T$ for leptons and $Q = (u, d)_L^T$ for quarks, reflecting the fact that the weak interaction couples only to left-handed particles. Right-handed fermions, such as e_R, u_R, d_R are singlets under $SU(2)_L$, which naturally explain the observed violation of parity in weak interactions. Each fermion field carries a define value of weak hypercharge Y, which is fixed by requiring consistency with observed electric charges. To construct a local gauge invariant theory gauge fields are introduced corresponding to each generator of the symmetry group. The $SU(2)_L$ symmetry introduces three-gauge fields W_μ^a with coupling constant g, while the $U(1)_Y$ symmetry introduces one gauge field B_μ with coupling g'. The ordinary derivative is replaced by the covariant derivative,

$$D_\mu = \partial_\mu - ig T^a W_\mu^a - ig' \frac{Y}{2} B_\mu \tag{6.36}$$

Which ensure invariance under local gauge transformations and automatically introduce interactions between fermions and gauge bosons. The fermionic part of the lagrangian is then written as,

$$\mathcal{L}_{fermion} = \bar{\psi}_L i\gamma^\mu D_\mu \psi_L + \bar{\psi}_R i\gamma^\mu D_\mu \psi_R \tag{6.37}$$

Eq. (6.37), describing both the free propagation of fermions and their electroweak interactions. The dynamics of the gauge fields themselves are governed by field strength tensor. For the non-abelian $SU(2)_L$ gauge fields, the field strength tensor is,

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon^{abc} W_\mu^b W_\nu^c \tag{6.38}$$

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Which contains self-interaction terms characteristics of non-Abelian gauge theories. For the Abelian $U(1)_Y$ field, the field strength is,

$$B_{\mu\nu} = \partial_\mu B_\mu - \partial_\nu B_\mu \tag{6.39}$$

The gauge field lagrangian is therefore,

$$\mathcal{L}_{gauge} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{6.40}$$

At this stage, the theory describes massless gauge boson, which contradicts experimental observations showing that the weak force is mediated by massive $W^\pm$ and Z^o bosons. Direct mass terms for gauge fields would violate gauge invariance, so masses must arise through a different mechanism. This is achieved through spontaneous symmetry breaking, implemented by introducing a complex scalar Higgs field in the form of an $SU(2)_L$ doublet,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{6.41}$$

The Higgs lagrangian is written as,

$$\mathcal{L}_H = (D_\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi) \tag{6.42}$$

Where the Higgs potential is chosen as,

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + (\Phi^\dagger \Phi)^2 \tag{6.43}$$

With $\mu^2 < 0$. This potential has a minimum at a nonzero value of the field, leading to spontaneous symmetry breaking. Minimizing the Higgs potential yields a vacuum expectation value,

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{6.44}$$

Where as $v = \sqrt{-\mu^2/\lambda}$. This vacuum state breaks the original $SU(2)_L \times U(1)_Y$ symmetry down to the electromagnetic subgroup $U(1)_{em}$. Substituting vacuum expectation value into the Higgs kinetic term generates mass terms for the gauge bosons. The charge gauge bosons are formed as,

$$W^\pm_\mu = \frac{1}{\sqrt{2}} (W^1_\mu \mp i W^2_\mu) \tag{6.45}$$

With masses, $m_W = 1/2 g v$. The neutral gauge fields W^3_μ and B_μ mix to form the physical Z_μ boson and the photon A_μ, according to,

$$\begin{pmatrix} Z_\mu \\ A_\pi \end{pmatrix} = \begin{pmatrix} cos\theta_W & -sin\theta_W \\ sin\theta_W & cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{6.46}$$

Where the Weinberg angle θ_W is defined by $tan\theta_W = g'/g$. The resulting masses are,

$$m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}, \quad m_\gamma = 0 \tag{6.47}$$

Showing that the photon remains massless while the weak interaction becomes short-ranged. Electric charge emerges naturally from the unbroken symmetry and is given by the relation,

$$Q = T^3 + \frac{Y}{2} \tag{6.48}$$

With the electromagnetic coupling constant defined as,

$$e = g\sin\theta_W = g'\cos\theta_W \tag{6.49}$$

Thus, electromagnetism appears as the residual unbroken gauge symmetry of the electroweak theory. Fermions masses are generated through Yukawa interaction between fermions and the Higgs field,

$$\mathcal{L}_Y = -y_f \bar{\psi}_L \Phi \psi_R + h.c \tag{6.50}$$

Eq. (6.50), represent the Yukawa lagrangian density. When the Higgs field acquires its vacuum expectation value, these interactions produce fermions mass terms,

$$m_f = \frac{y_f v}{\sqrt{2}} \tag{6.51}$$

Eq. (6.51) demonstrating that the fermions masses arise from their coupling to the Higgs field, with different Yukawa coupling explaining the observed mass hierarchy. After symmetry breaking the Higgs field can be written as,

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{6.52}$$

Where the h(x) represents a physical scalar excitation, the Higgs boson, with masses $m_H = \sqrt{2\lambda}v$. The complete electroweak lagrangian density is therefore the sum of fermions, gauge, Higgs and Yukawa terms,

$$\mathcal{L}_{EW} = -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \sum_f \bar{\psi}_L i\gamma^\mu \left(\partial_\mu - ig\frac{\tau^2}{2}W^a_\mu - ig'\frac{Y}{2}B_\mu\right)\psi_f +$$
$$(D_\mu\Phi)^\dagger(D_\mu\Phi) - V(\Phi) + y_f\bar{\psi}_L\Phi\psi_R + h.c \tag{6.53}$$

Eq. (6.53), provides a unified and experimentally verified description of electromagnetic and weak interaction. At high energies, the symmetry is restored and the forces are unified, while at low energies the symmetry is broken, giving rise to massive weak bosons, a massless photon.

Quantum chromodynamics (QCD) is the relativistic quantum filed theory that describes the strong interaction between quarks and gluons, which are the fundamental components of hadrons such as protons, neutron and mesons. This theory is based on the principle of local gauge invariance under non-abelian symmetry group $SU(3)_{color}$. Unlike QED, where the gauge group is abelian and gauge boson is electrically neutral, QCD possesses a richer structure because the gluons

themselves carry color charge, leading to self-interaction that dominate the dynamics of the strong force. Quarks are described by Dirac spinor field $\psi_f(x)$ where the index f labels the quarks flavor, and each quark field transform as a triplet under the color group. In the absence interactions, the dynamics of a quark field of mass m_f is governed by the free Dirac lagrangian,

$$\mathcal{L}_o = \bar{\psi}_f(i\gamma^\mu\partial_\mu - m_f)\psi_f \tag{6.54}$$

This lagrangian is invariant under global $SU(3)$ color transformations, but it fails to remain invariant when the symmetry is promoted to a local one, because the ordinary derivative ∂_μ acts on the spacetime dependent transformation parameters. To restore local gauge invariance, eight-gauge fields $G_\mu^a(x)$, known as gluon fields, are introduced, corresponding to the eight generators $T^a = \lambda^a/2$ of the $SU(3)$ group, where λ^a are the Gell-Mann matrices. The ordinary derivatives are replaced by the covariant derivative,

$$D_\mu = \partial_\mu - ig_s T^a W_\mu^a \tag{6.55}$$

With g_S denoting the strong coupling constant. With this replacement the quark part of the lagrangian becomes,

$$\mathcal{L}_{quark} = \bar{\psi}_f(i\gamma^\mu D_\mu - m_f)\psi_f \tag{6.56}$$

Which is invariant under local $SU(3)$ color transformations and naturally contains the interaction between quarks and gluons. The dynamics of the gluon fields themselves is determined by the non-abelian field strength tensor, which arises from the commutator of covariant derivatives,

$$[D_\mu, D_\nu] = -ig_s T^a G_{\mu\nu}^a \tag{6.57}$$

Explicitly, the gluon field strength tensor is given by,

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c \tag{6.58}$$

Where f^{abc} are the structure constants of the $SU(3)$ algebra. The presence of the quadratic term in the gauge fields reflects the non-abelian nature of QCD and leads to three gluon and four gluon self-interaction vertices, which have no analogue in QED. The gauge invariant kinetic term for the gluon fields is constructed as,

$$\mathcal{L}_{gluon} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} \tag{6.59}$$

This term accounts for the propagation of gluons as well as their self-interactions. Combining the quarks and gluon contributions and summing over all quark flavors yields the full QCD, lagrangian density,

$$\mathcal{L}_{QCD} = \sum_f \bar{\psi}_f(i\gamma^\mu D_\mu - m_f)\psi_f - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} \tag{6.60}$$

This single lagrangian summarize all strong-interaction phenomena. From it follow the quark-gluon interaction responsible for binding quarks inside hadrons, as well as the self-interaction gluon dynamics that give rise to confinement and asymptotic freedom. At high energies, the effective strong coupling decreases logarithmically allowing quarks to behave as nearly free particles, while at low energies the coupling become strong confining quarks and gluons into color-neutral bound states. A Feynman diagram is seen as in figure 6.4, for the strong force shows how quarks interact by exchanging force carrying particles called gluons. In the diagram quarks are represented by straight lines, while the strong interaction is mediated by curly lines corresponding to the gluon. For example, inside a proton or neutron, quarks continuously exchange gluons, which carry color charge and bind the quarks together. Unlike other forces, gluons can also interact with each other, making the strong force very powerful at short distances and responsible for holding atomic nuclei together. Thus QCD, lagrangian provides a complete and fundamental description of strong force within the frame work of quantum field theory.
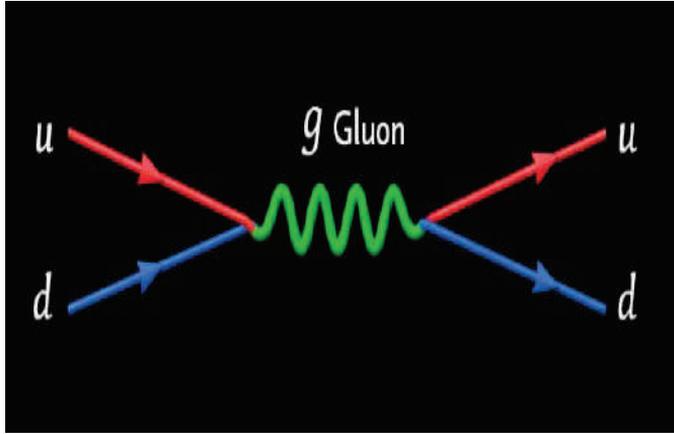
Figure 6.4: Representation of Feynman diagram of strong interaction mediated by gluon.

The standard model of particle physics is a relativistic quantum field theory that provides a unified frame work for electromagnetism, weak and strong interactions. The lagrangian density of the standard model of physics is the sum of lagrangian densities of gauge bosons, fermions, Higgs and Yukawa,

$$\mathcal{L}_{SM} = -\frac{1}{4}G_{\mu\nu}^{a}G^{a\mu\nu} - \frac{1}{4}W_{\mu\nu}^{i}W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \Sigma_{generation}(\bar{Q}_{L}i\gamma^{\mu}D_{\mu}Q_{L} + \bar{u}_{R}i\gamma^{\mu}D_{\mu}u_{R} +$$
$$\bar{d}_{R}i\gamma^{\mu}D_{\mu}d_{R} + \bar{L}_{L}i\gamma^{\mu}D_{\mu}L_{L} + \bar{e}_{R}i\gamma^{\mu}D_{\mu}e_{R}) + (D_{\mu}\Phi)^{\dagger}(D^{\mu}\Phi) - \mu^{2}\Phi^{\dagger}\Phi - \lambda(\Phi^{\dagger}\Phi)^{2} -$$
$$(\bar{L}_{L}Y_{e}\Phi e_{R} + \bar{Q}_{L}Y_{d}\Phi d_{R} + \bar{Q}_{L}Y_{u}\tilde{\Phi}u_{R} + h.c \qquad (6.61)$$

Eq. (6.61) is a complex mathematical lagrangian that summarizes all known fundamental particles (quarks, leptons, force carriers like photons, gluons, and W/Z bosons) and their interactions describing the electromagnetic, strong and weak forces plus the Higgs mechanism for mass,

essentially dictating how universe's building blocks behaves and exchange energy between kinetic and potential to quantum fields. The standard model of particle physics is seen as in figure 6.5, classified all known fundamental particle into fermions, consist of quarks (up, down, charm, strange, top and bottom), carrying fractional electric charge and color charge interacting via strong force, and leptons (electron, muon, tau, and their neutrinos) which are color neutral and participate in weak and electromagnetic interactions. Force carrier bosons mediate interactions: the photon (γ) for electromagnetism, $W^{\pm}$ and Z^o for the weak force, gluons (g) for strong force and Higgs boson (H) generates mass through spontaneous symmetry breaking. Electromagnetic interaction occurs between charged particle via photon exchange, weak interactions, mediated by massive W/Z bosons, change particle flavor and govern processes like beta decay, strong force binds quarks into hadrons via gluon exchange. The Higgs field couples to massive particles giving fermions and W/Z bosons with mass. Together these interactions explain the structure of matter, from quarks forming protons and neutrons, electron forming atoms, to neutrinos mediating weak processes in stars. As a result, standard model explains all forces, excluding gravity particles and their interaction at quantum level.
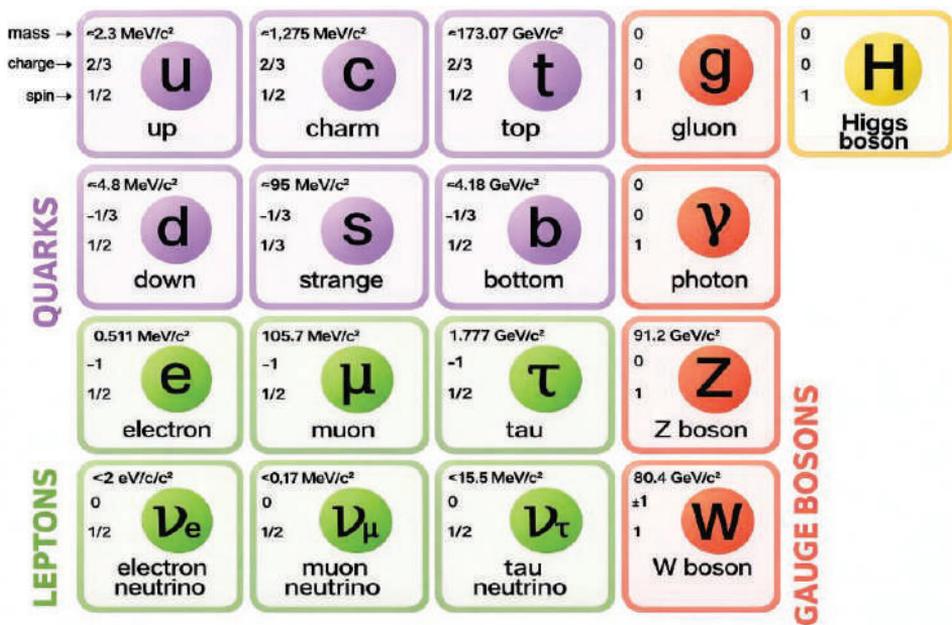

Figure 6.5: Standard model of particle physics.

Chapter # 07

Loop quantum gravity

The standard model of physics explains all fundamental forces but did not explain the gravity at the quantum level. The problem of constructing a quantum theory of gravity arises from the conceptual and mathematical incompatibility between general relativity and quantum field theory. General relativity describes gravity as the dynamics of spacetime geometry determine in the spacetime metric $g_{\mu\nu}$, while quantum field theory treats physical systems as operators acting on a Hilbert space, usually defined on a fixed background spacetime. Attempts to quantize gravity perturbatively by expanding the metric around flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7.1}$$

Leads to ultraviolet divergences that cannot be renormalized. This failure indicates that gravity must be quantized in a fundamentally different manner, one that respects the background independence intrinsic to general theory of relativity. Loop quantum gravity (LQG), provides such a framework by directly quantizing spacetime geometry itself. The classical dynamics of the gravity are derived from Einstein-Helbert action,

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \tag{7.2}$$

Where R is the Ricci scalar curvature and G is the Newton gravitational constant. Variation of this action with respect to the metric tensor $g_{\mu\nu}$, yields Einstein's filed equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{7.3}$$

To quantize gravity canonically spacetime is foliated into a family of spatial hyperspaces labeled by a time parameter, leading to 3+1 decomposition. In this frame work the spatial metric q_{ab} becomes the configuration variable and its conjugate momentum is related to the extrinsic curvature. A central step towards LQG, is the reformulation of general relativity in terms of tetrads. Tetrads are a set of four orthonormal basis vectors defined at each point in spacetime that represent the gravitational field acting as the bridge between the metric tensor and the gauge theoretic variables (Ashtekar variables) used in quantization. The spacetime metric is expressed as,

$$g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ} \tag{7.4}$$

And on spatial hypersurface the metric becomes,

$$q_{ab} = e_a^i e_b^j \delta_{ij} \tag{7.5}$$

Where e_a^i are triads and δ_{ij} is the delta function. This formulation allows gravity to be written as a gauge theory. Ashtekar introduced new variables that transform gravity into a theory resembling

$SU(2)$ Yang-Mills theory. The fundamental variables of loop quantum gravity are the Ashtekar-Barbero connection,

$$A_a^i = \Gamma_a^i + \gamma K_a^i \tag{7.6}$$

And the densitized traid,

$$E_i^a = \sqrt{q} e_i^a \tag{7.7}$$

Here Γ_a^i is the spin connection determined by the triads, K_a^i is the extrinsic curvature and γ is the Barbero-Immirzi parameters. The variables satisfy the canonical Poisson bracket,

$$\{A_a^i(x), E_j^b(y)\} = 8\pi G \gamma \delta_a^b \delta_j^i \delta^3(x-y) \tag{7.8}$$

Because the general relativity is a generally covariant theory, the dynamics are entirely fixed in constrains. The Gauss constrains,

$$G_i = D_a E_i^a = 0 \tag{7.9}$$

Ensures $SU(2)$ gauge invariance. The diffeomorphism constraint,

$$C_a = E_i^a F_{ab}^i = 0 \tag{7.10}$$

Generates spatial coordinate transformations. The Hamiltonian constraint,

$$\mathcal{H} = \frac{E_i^a E_j^b}{\sqrt{detE}} (\varepsilon_k^{ij} F_{ab}^k - 2(1+\gamma^2) K_a^i K_b^i = 0 \tag{7.11}$$

Contains the full dynamical content of Einstein's equations. Here F_{ab}^i is the curvature of the Ashtekar connections,

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \varepsilon_{jk}^i A_a^j A_b^k \tag{7.12}$$

The quantization of gravity in LQG does not promote the connection and triad directly to operators. Instead, instead triad and operators the fundamental variables of loop quantum gravity are holonomies and fluxes is seen as in figure 7.1 (a), and figure 7.1 (b). Holonomies represents how a gravitational connection changes when it is transported along a loop. Mathematically a holonomy is the path-ordered exponential of the connection along a curve. Physically it is determined information about curvature by measuring how vectors rotate when moved around a closed loop. In LQG, holonomies are associated with the edges of spin networks. The holonomy of the connection along an oriented curve e, is defined as,

$$h_e[A] = p \exp\left(\int_e A_a^i \tau_i dx^a\right) \tag{7.13}$$

Where τ_i are generators of the $SU(2)$ algebra.

Fluxes are related to the triad and represent the flow of geometric information through a surface. The are defined as surface integral of the triad field. Flux operators are associated with the surfaces intersecting spin network edges and are responsible for giving discrete eigenvalues of area. The flux of the densitized triad through a surface S is given by,

$$E_i(S) = \int_S \varepsilon_{abc} E_i^a dx^b \wedge dx^c \tag{7.14}$$

These variables are well-defined without reference to a background metric and form the basic algebra of loop quantum gravity. Quantum states of geometry are represented by spin network states.
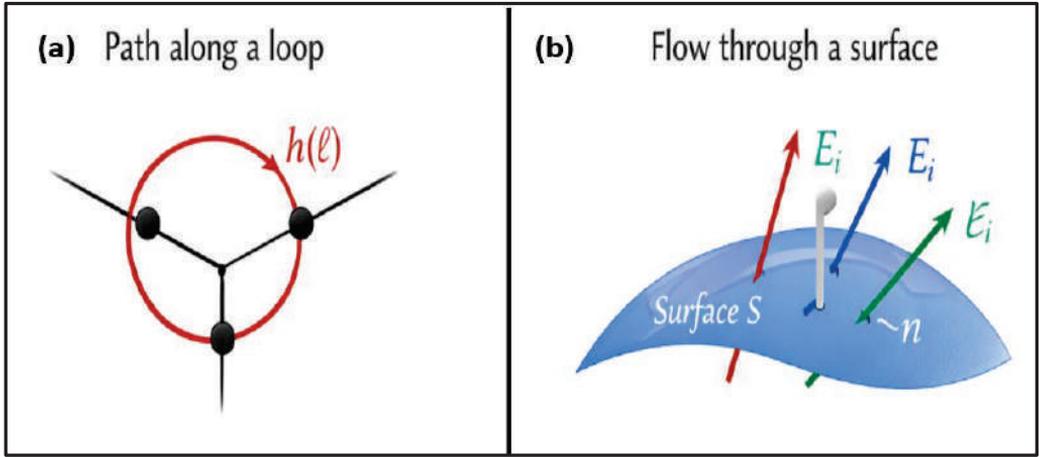

Figure 7.1: Representation of (a) Holonomies and (b) Fluxes.

A spin network consists of a graph Γ whose edges are labeled by $SU(2)$ representations j_e and whose vertices are labeled by intertwiners i_v. A generic quantum state is written as,

$$|\Psi\rangle = |\Gamma, j_e, i_v\rangle \tag{7.15}$$

Spin network represented in figure 7.2 (a), is a quantum state that describes the discrete structure of space at plank scale. It is a graph made of edges and nodes, where each edge is labeled by a quantum number representing an irreducible representation of SU(2), and each node represents an intertwiner that connects these representations consistently. Physically the edges correspond to quantize units of area and nodes correspond to quantized units of volume. One of the most consequences of loop quantum gravity is the quantization of geometric observables. The area operator has discrete eigenvalues determines by spin label on the edges of a spin network that intersect surface in seen as in figure 7.2 (b), associated with a two-dimensional surface S, acts on a spin network states as,

$$\hat{A}(S) = 8\pi\gamma l_p^2 \sum_{p \in S \cap \Gamma} \sqrt{j_p(j_p + 1)} \tag{7.16}$$

Where $l_p = \sqrt{G\hbar}$ is the plank length and the sum runs over holes where the spin network intersects the surface. The spectrum of this operator is discrete implying that area comes in fundamental quanta. The volume operator has discrete eigenvalues associated with the nodes (intertwiner) of the spin network is seen in figure 7.2 (c). The volume operator of a spatial region R is given by,

$$\hat{V}(R) = \Sigma_{v \in R} \sqrt{\left| \frac{1}{48} \varepsilon^{ijk} \varepsilon_{abc} E_i^a E_j^b E_k^c \right|} \tag{7.17}$$
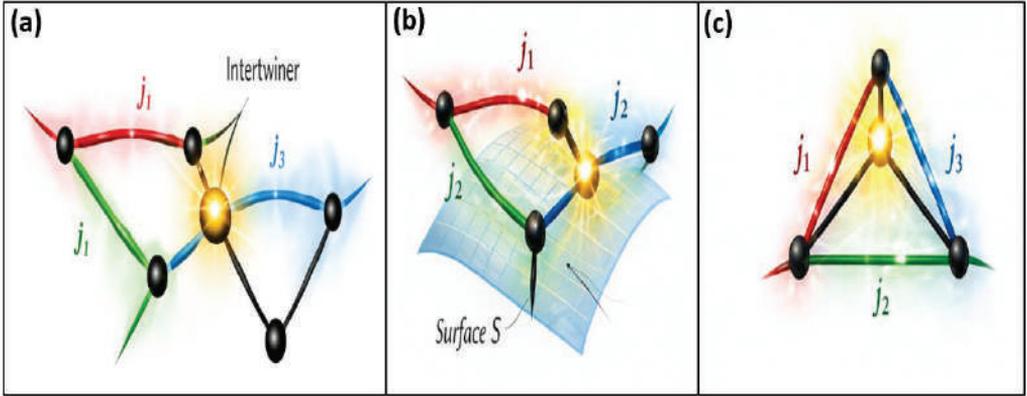

Figure 7.2: Representation of (a) Spin network (b) quantization of area operator and (c) quantization of volume operator.

With eigenvalues determined by the intertwiners at the nodes of the spin network. Length operator can also be defined and exhibits discrete spectra of the form,

$$\hat{L} = lp\sqrt{j(j+1)} \tag{7.18}$$

Using tetrads e_μ^I and the spin connection ω_μ^{IJ} the palatini action is expressed as,

$$S_{palatini}[e, \omega] = \frac{1}{16\pi G} \int \varepsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}[\omega] \tag{7.19}$$

Where $F^{KL}[\omega] = d\omega^{KL} + \omega_M^K \wedge \omega^{ML}$ is the curvature of the spin connection, and ε_{IJKL} is the totally antisymmetric tensor. This formulation treats gravity as an $SU(2)$ gauge theory when restricted to 3+1 dimensions. The classical action in terms of Ashtekar-Barbero variables,

$$S[A, E] = \frac{1}{8\pi G\gamma} \int dt \int_\Sigma dx^3 E_i^a \dot{A}_a^i - N^a C_a - N\mathcal{H} - \lambda^i G_i \tag{7.20}$$

Where γ is Barbero-Immirzi parameters, N, N^a are the lapse and shift functions, λ^i are lagrange multipliers for SU(2) Gauss constraint, G_i is the Gauss constraint, C_a is the diffeomorphism

constraint and $\mathcal{H}$ is the Hamiltonian constraint. The action of LQG, in terms of holonomy and flux representation,

$$S_{LQG}[h, E] = \int dt \sum_e Tr \left(E_i(S_e) \dot{h}_e h_e^{-1} \right) - \sum_{constaints} \lambda C \qquad (7.21)$$

Varying the action of the loop quantum gravity with respect to canonical variables yields the Einstein quantum equation,

$$\hat{\mathcal{H}} |\Psi\rangle = 0 \qquad (7.22)$$

Eq. (7.22), represent the quantum version of einstein field equation in loop quantum gravity and $|\Psi\rangle$ is the spin network state. This equation tells us that geometry is quantized, and length, are and volume have a discrete-spectra. This is the loop version of the Wheller-DeWitt equation. Where the Hamiltonian constrain operator is,

$$\hat{\mathcal{H}} = \frac{1}{16\pi G} \int dx^3 \varepsilon^{ijk} Tr(\hat{F}_{ab}^i \, \hat{E}_j^a \hat{E}_k^b) \qquad (7.23)$$

Where $\hat{F}_{ab}^i$ represent the curvature of the quantum connection, operator acts on spin network, and curvature is represented by holonomies around loops.

In classical general relativity the big bang is a singularity, as we trace the universe backward in time the scale factor $a(t) \to 0$, and physical quantities like energy density ρ and curvature R diverge. Mathematically this can be seen from Friedmann equation for a flat universe,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho \qquad (7.24)$$

As $a \to 0$, $\rho \to \infty$ leading to a breakdown of classical physics. Loop quantum gravity through its quantization of spacetime geometry fundamentally changes this behavior. In LQG, spatial geometry is discrete, areas, volume and length have quantized spectra. This discreteness implies that there is a minimum nonzero volume of space preventing the scale factor from reaching zero. In LQG, the Friedmann equation is modified by quantum geometry corrections,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_C}\right) \qquad (7.25)$$

Where $\rho_C \sim \rho_{planck}$ is the critical energy density at which quantum gravity effects become significant. The key point in Eq. (7.25) is the correction term $1 - \rho/\rho_c$. When $\rho \ll \rho_c$ the classical Friedmann equation is recovered. However, as $\rho \to \rho_c$ the Hubble parameter $\dot{a}/a \to 0$, uncertain contraction and preventing the singularity. Instead of a big bang singularity, the universe undergoes a quantum bounce: The collapsing phase reaches a minimum volume and then expand again is seen as in figure 7.3. Physically the bounce arises because the quantum discreteness of spacetime provides a repulsive force at plank scale. This repulsion is purely geometric and does not require exotic matter. The bounce naturally replaces the singularity with a finite well plank scale universe.
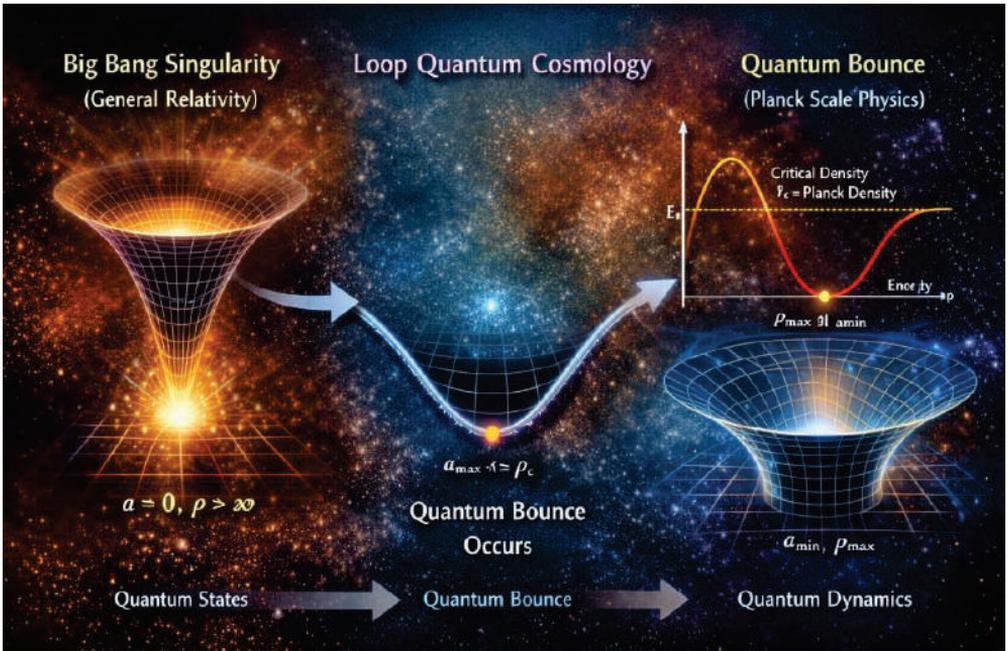
Figure 7.3: Quantum bounce which replaces classical singularity.

Chapter # 08

String theory

Loop quantum gravity successfully quantize gravity through spacetime geometry. In LQG, areas, and volumes become discrete and spacetime is represented by spin networks rather than a smooth continuum. This approach avoids perturbative infinities and provides a mathematically consistent quantization of gravity. However, LQG, only quantize gravity it did not explain the origin of gravity. It does not include the gauge fields or matter content of standard model of physics. LQG, is only the quantized version of gravity not unified theory of physics. On the other hand, string theory is constructed from the observation that the ultraviolet inconsistencies of quantum gravity originate from assumption that fundamental particles are pointlike. In quantum field theory interactions occur at the spacetime points producing divergence that become uncontrollable when gravity is included. String theory resolves this problem by postulating that the fundamental objects are one-dimensional extended objects called string is seen in figure 8.1.
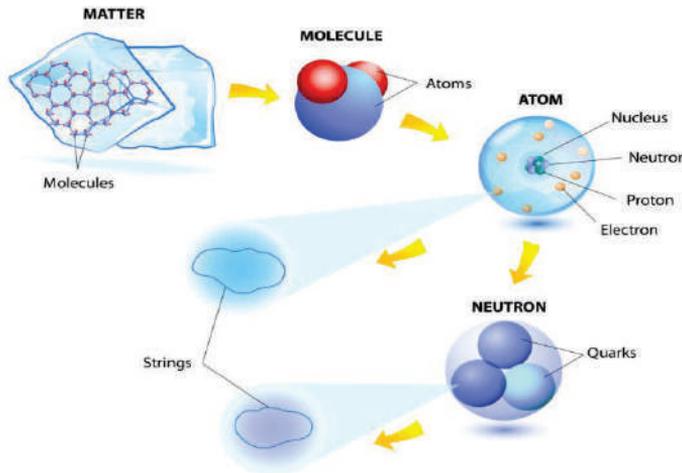
Figure 8.1: Graphical representation of string theory.

The evolution of a strings in spacetime traces out a two-dimensional surface known as worldsheet is seen as in figure 8.2 (a). Mathematically, the worldsheet is parameterized by two coordinates: The time like τ describing the evolution of the string, and the spacelike parameter σ describing points along the string. The embedding the worldsheet in spacetime is given by $X^\mu(\tau, \sigma)$ which describe the spacetime trajectory of the string. Quantization of the worldsheet leads to the spectrum of string excitations, giving rise to particles with different spins and masses. The function of space time is given, by Eq. (7.1),

$$X^\mu(\tau, \sigma), \qquad \mu = 0, 1, \dots, D-1 \tag{7.1}$$

Figure 8.2: Representation of (a) worldsheet and (b) D-brane for closed and open strings.

D-branes (Dirichlet-branes) are extended objects in string theory on which open strings can end. The end points of open strings satisfy Dirichlet boundary conditions along direction perpendicular to the D-brane and Neumann boundary conditions along direction parallel to it. Dp-brane is seen in figure 8.2 (b), is a p-dimensional hypersurface in spacetime, meaning it extends along p s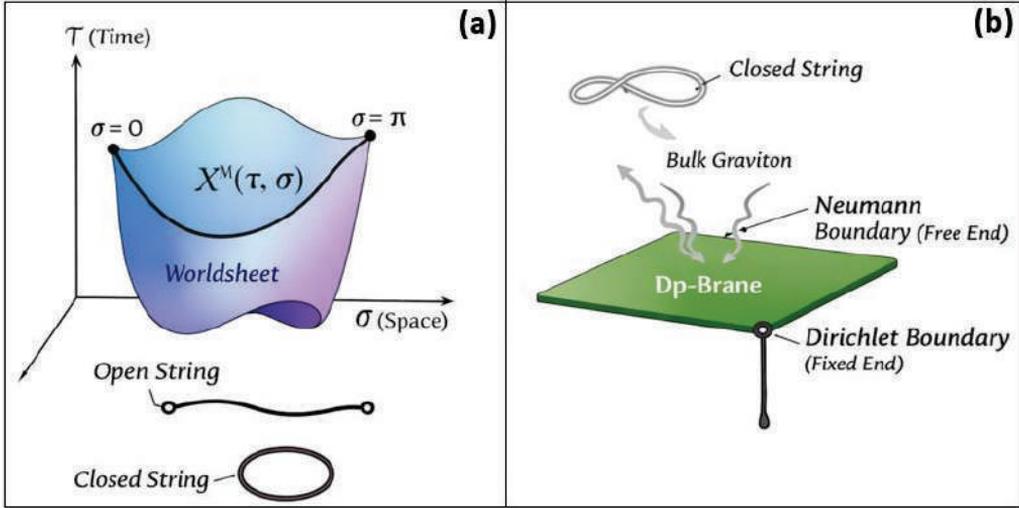pecial directions. D-branes play a critical role in non-perturbative string theory, providing a framework to study gauge theory, black holes and string dualities. The excitations of open strings ending on D-branes corresponding to gauge fields living on the brane worldvolume, linking D-branes to low energy effective field theories. Additionally, D-branes can interact with closed strings, serving as sources for gravitational and other fields in the spacetime. The dynamics of a relativistic string are governed by an action proportional to the area of the worldsheet. The Nambu-Goto action is,

$$S_{NG} = -T \int d\tau d\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X_\mu)} \tag{7.2}$$

Where T represents the string tension. For quantization it is convenient to introduce an independent worldsheet metric h_{ab}, leading to the Polyakov action,

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu \tag{7.3}$$

This action is invariant under worldsheet diffeomorphism and Weyl transformations. Varying the action with respect to X^μ gives the equations of motion,

$$\partial_a(\sqrt{-h} h^{ab} \partial_b X^\mu) = 0 \tag{7.4}$$

Choosing conformal gauge $h_{ab} = e^{2\phi}\eta_{ab}$, Eq. (7.4) reduces to two-dimensional wave equation,

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0 \tag{7.5}$$

Varying the action with respect to the worldsheet metric yields the vanishing of the stress energy tensor,

$$T_{ab} = 0 \tag{7.6}$$

Which leads to the Virasoro constraints,

$$(\partial_\tau X \pm \partial_\sigma)^2 = 0 \tag{7.7}$$

Closed string satisfies periodic boundary conditions,

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) \tag{7.8}$$

While open string satisfies Neumann or Dirichlet boundary conditions,

$$\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0 \ \ or \ X^\mu|_{\sigma=0,\pi} = const \tag{7.9}$$

The general solution for closed strings is,

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \Sigma_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right) \tag{7.10}$$

While for open strings it is,

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \Sigma_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-in\tau}\cos(n\sigma)\right) \tag{7.11}$$

Canonical quantization imposes the commutation relations for closed and open strings,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,}0\eta^{\mu\nu} \tag{7.12}$$

$$[\alpha'^\mu_m, \alpha'^\nu_n] = m\delta_{m+n,}0\eta^{\mu\nu} \tag{7.13}$$

The Virasoro generators are,

$$L_n = \frac{1}{2}\Sigma_m \, \alpha_{n-m} \cdot \alpha_m \tag{7.14}$$

$$\tilde{L}_n = \frac{1}{2}\Sigma_m \, \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m \tag{7.15}$$

Satisfy the Virasoro algebra,

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,}0 \tag{7.16}$$

Quantum consistency requires vanishing of the conformal anomaly, fixing the critical dimensions. Physical states satisfy,

$$L_n|\psi\rangle = 0 \ (n > 0) \tag{7.17}$$

$$(L_0 - a)|\psi\rangle = 0 \tag{7.18}$$

The resulting mass spectra are,

$$M^2_{open} = \frac{1}{\alpha'}(N - 1) \tag{7.19}$$

$$M^2_{closed} = \frac{4}{\alpha'}(N + \tilde{N} - 2) \tag{7.20}$$

The massless open-string state $\alpha^\mu_{-1}|0\rangle$ corresponds to a gauge field A_μ, while the massless closed-string state $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0\rangle$ decomposes into $g_{\mu\nu}, B_{\mu\nu}, \emptyset$, representing the gravitons, antisymmetric tensor field and dilation.

In string theory closed strings correspond to gravitons (spin-2) i.e. gravity, while open strings correspond to gauge bosons (spin-1) i.e. Yang-Mills forces like electromagnetism, weak and strong interactions. So, gravity and gauge forces emerge naturally from different vibrational modes of strings. Open strings end on Dp-brane producing gauge fields living on the brane. The low energy effective action is the Dirac-Born-Infeld (DBI) action,

$$S_{DBI} = -T_P \int d^{p+1}\xi e^{-\phi} Tr \sqrt{-\det\left(P[g]_{ab} + 2\pi\alpha' F_{ab} - B_{ab}\right)} \tag{7.21}$$

Where T_P represent Dp-brane tension, ξ represent worldvolume coordinates, $P[g]_{ab}$ represent pullback of bulk metric onto brane, F_{ab} represent field strength of the gauge field on the brane, B_{ab} represent pullback of the bulk 2-form $B_{\mu\nu}$ and $e^{-\phi}$ represent coupling to the dilation. Expanding DBI action at low energies gives the familiar Yang-Mills action,

$$S_{YM} = -\frac{1}{4g^2_{YM}} \int d^{p+1}\xi Tr F_{ab} F^{ab} \tag{7.22}$$

The low energy effective action for closed strings is the Einstein-Hilbert action with extra fields,

$$S_g = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g}\left(R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\right) \tag{7.23}$$

Where represent the Ricci scalar, $H_{\mu\nu\rho}$ represent the antisymmetric tensor field, and Φ represent dilation, controls string coupling. Both the gauge field action and gravity action arises from the same string theory. Strings can split and join. For example, two open strings join to form a closed string. This automatically links gauge interactions (open strings) to gravity (closed strings).

Mathematically this is set in string scattering amplitudes which include both gauge bosons and gravitons. The scattering amplitude relation,

$$A_{closed} \sim \left(A_{open}\right)^2 \tag{7.24}$$

This is the famous KLT relation (Kawai-Lewellen-Tye). It is how that graviton amplitudes are essentially squares of gauge boson amplitudes unifying gravity with gauge theory at the level of string interactions. The coupling of gravity with gauge theory is seen as in figure 8.3, through Ramond-Ramond (RR) couplings described by the Chern-Simons (CS) action,

$$S_{cs} = \mu_p \int_{W_{p+1}} C \wedge e^{2\pi\alpha' F + B} \tag{7.25}$$

Where W_{p+1} denotes the (p+1) dimensional worldvolume of the Dp-brane, μ_p represent the Dp-brane charge, $C = \sum_q C_q$ represents the sum over all Ramond-Ramond q-form potentials of the closed-string sector, $F = dA + A \wedge A$ is the non-Abelian Yang-Mills field strength arising from open strings ending on the brane, and B is the Neveu-Schwarz two-form field of the closed string sector.

Figure 8.3: Representations of Strings coupling gravity with gauge fields.

String theory modifies the classical cosmology by replacing point particles with one-dimensional strings of fundamental length $l_s = \sqrt{\alpha'}$, which implies a minimum resolvable distance $\Delta x \geq l_s$. The fundamental dynamics are described by the Polyakov action Eq. (7.3). So, interactions are spread over a worldsheet rather than localized at a point. At energies well below the string scale, closed strings give rise to an affective gravitational theory whose action is,

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \alpha' R^2 + \cdots\right) \tag{7.26}$$

Where ϕ is the dilation and α' is the string corrections. Near the big bang when curvature becomes large these higher order terms dominate and invalidate the classical Einstein equations. Upon dimensional reduction to a four-dimensional FRW background the cosmological part of the action becomes,

$$S = \int dt a^3 e^{-2\phi}(-6H^2 + 4\dot{\phi}^2 + \alpha' H^4 + \cdots \tag{7.27}$$

Varying the above equation with respect to the metric yields a string-Modified Friedmann equation,

$$H^2 - 2H\dot{\phi} + \dot{\phi}^2 + \alpha' H^4 = \frac{8\pi G}{3}\rho \tag{7.28}$$

While varying with respect to dilation gives,

$$\ddot{\phi} - 3H\dot{\phi} + \dot{\phi}^2 + \frac{1}{2}\alpha' H^4 = 0 \tag{7.29}$$

These equations imply that the Hubble parameter is bounded,

$$H^2 \leq \frac{1}{\alpha'} \tag{7.30}$$

So, the curvature invariants remain finite and the cosmological singularity is removed. A concrete realization of this behavior appears in pre-big bang cosmology, where a consistent solution in the string frame is,

$$a(t) \sim (-t)^{-1\sqrt{3}} \qquad t < 0 \tag{7.31}$$

$$\phi(t) \sim -\ln(-t) \qquad t < 0 \tag{7.32}$$

At t approaches the string scale $t \sim -\sqrt{\alpha'}$, the universe enters a high curvature but finite string phase and then smoothly evolves into the standard hot big bang regime. An additional mechanism preventing a singular origin is T-duality, under which physics is invariant when the radius of a compact dimension transforms as,

$$R \leftrightarrow \frac{\alpha'}{R} \tag{7.33}$$

Because a vanishing radius is equivalent to a large one, the scale factor never reaches a physically meaningful zero. Consistently, in strings gas cosmology the temperature is bounded by the Hagedorn temperature,

$$T \leq T_H \sim \frac{1}{\sqrt{\alpha'}} \tag{7.34}$$

So thermal quantities also remain finite. In this way, string theory replaces the classical big bang singularity with finite, duality-invariant quantum phase governed by extended strings and higher curvature corrections. The big bang is no longer the beginning of spacetime but a smooth transition from older universe to the new universe is seen as in figure 8.4.
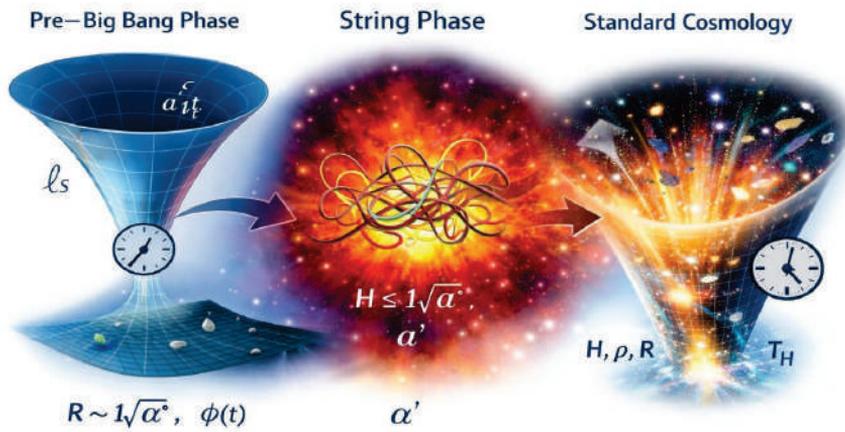
Figure 8.5: Big bang is no the beginning it is the transition of universe from one phase to another phase.

Chapter # 09

Superstrings theory

String theory in its original bosonic form describes fundamental particles as one-dimensional vibrating strings naturally incorporating gravity through closed strings. However, it faces problems: it lacks fermions, contains tachyonic instability and requires 26 spacetime dimensions, indicating string theory is physically incomplete. Superstring theory resolves these limitations by incorporating supersymmetry, which introduces a symmetry between bosons and fermions, stabilizes the vacuum by removing tachyons and reduces the critical dimension to 10, allowing a consistent description of matter and forces. Consequently, superstring theory not only retains the ability of strings to unify gravity and gauge interactions but also provides a stable realistic framework capable of describing all fundamental particles making it a practical candidate for a unified theory of nature. Superstring theory consists of five formulations is seen as in figure 9.1, each with unique features, yet all sharing ten-dimensional spacetime and supersymmetry (a symmetry that relates bosons and fermions, transforming one into the other). Type I superstring theory is distinguished by the presence of both open and closed strings. Open string has endpoints than can attach to D-branes, naturally generating non-Abelian gauge interactions, while closed strings include the graviton, incorporating gravity. The type I theory is chiral with $N = 1$ supersymmetry in ten-dimension, and its gauge group must be $SO(32)$ to cancel anomalies. Its low energy limit is described by ten-dimensional supergravity (gravity plus supersymmetry) coupled to super Yand-Mills fields. The low energy action for type I superstring theory is,

$$S = \frac{1}{2k_{10}^2} \int d^{10}x \sqrt{-g} \left[e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) - \frac{\alpha'}{4} e^{-\phi} Tr\left(F_{\mu\nu} F^{\mu\nu} \right) \right] \quad (9.1)$$

Where $g_{\mu\nu}$ represents graviton, ϕ represents dilation, $H_3 = dB_2 - \omega_{YM}$ represent Green-Schwarz modification, while $F_{\mu\nu}$ represents SO(32) gauge field.

Type IIA superstring theory contains only closed strings and is non-chiral, possessing $N = 2$ supersymmetry in ten-dimensions. Its massless spectrum includes the graviton, dilation and an asymmetric 2-form and a series of Ramond-Ramond fields. Type IIA supports D-branes of even dimensions, including point like D0-branes. Its low energy limit reduces to type IIA supergravity, and it serves as an important theory in the superstring theories. The action for the Type IIA superstring theory is,

$$S = \frac{1}{2k_{10}^2} \int d^{10}x \sqrt{-g} \left[e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_3^2 \right) - \frac{1}{4} F_2^2 - \frac{1}{48} F_4^2 \right] - \frac{1}{4k_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \quad (9.2)$$

This action contains gravity, dilation, the NS-NS-2 form (strings), R-R fields (D-branes), with kinetic and topological (Chern-Simon) terms.

Type IIB superstring theory is a ten-dimensional closed oriented with chiral $N = 2$ supersymmetry, meaning that both left and right moving fermions have same chirality, giving a total of 32

supercharge. Strings in type IIB are closed and there are no open strings or perturbative gauge fields, though gauge interactions arise non-perturbatively through D-branes. The action for type IIB superstring theory is,

$$S = \frac{1}{2k_{10}^2} \int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H_3^2\right) - \frac{1}{2}(\partial C_0)^2 \; \frac{1}{12}F_3^2 - \frac{1}{480}\tilde{F}_5^2\right] - \frac{1}{4k_{10}^2}\int C_4 \wedge$$
$$H_3 \wedge F_3 \qquad\qquad (9.3)$$

Where $e^{-2\phi}R$ represent the Einstein-Hilbert term, $4e^{-2\phi}(\nabla\phi)^2$ represent the kinetic energy of the dilation field controlling the string coupling, $e^{-2\phi}\frac{1}{12}H_3^2$ represents the energy of the antisymmetric 2-form B$_2$, the R-R fields contribute $\frac{1}{2}(\partial C_0)^2$ for the scalar axion C$_0$, the Chern-Simons term $\int C_4 \wedge H_3 \wedge F_3$ represents topological interactions between NS-NS and R-R fields and ensures consistent coupling to D-branes.
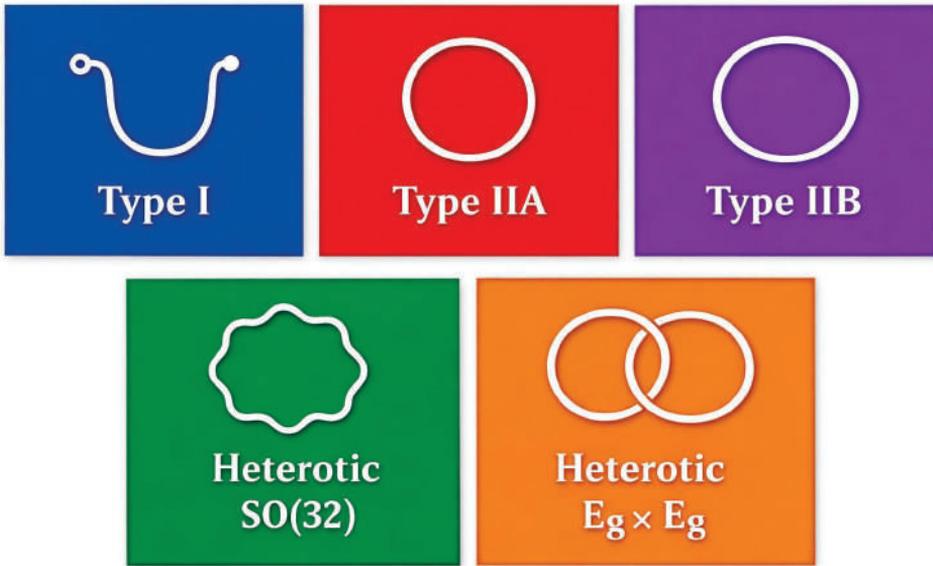
Figure 9.1: Different types of superstring theories i.e. Type I, Type IIA, Type IIB, Heterotic SO(32) and Heterotic Eg×Eg respectively and representation of strings of each type.

Heterotic superstring theory represents a remarkable unification between supersymmetric strings with non-Abelian gauge symmetries in ten-dimensions. Unlike conventional superstrings the heterotic construction is inherently asymmetric: the right moving sector of the string behaves like a superstring, providing spacetime supersymmetry and giving rise to the gravitational fields, the gravitons, the antisymmetric tensor, and the dilation while the left moving sector behaves like a bosonic string in 26 dimensions. The extra 16 left moving dimensions are compactified on a

carefully chosen lattice, which generates a consistent gauge symmetry and ensures the theory is free of anomalies. There exist only two anomaly-free choices for this left moving lattice, corresponding to two distinct heterotic superstring theories. The first is the SO(32) heterotic string, in which compactification lattice produces gauge bosons in the adjoint representation of SO(32). This theory naturally unifies gravity with a large non-Abelian gauge group and is notable for its S-duality connection to the type I superstring, linking strong and weak coupling regimes. The second the $E_8 \times E_8$ heterotic superstring theory where the 16-dimensional lattice splits into two E_8 lattices, giving rise to gauge fields in the adjoint representation of the $E_8 \times E_8$. The action for both SO(32) and $E_8 \times E_8$ heterotic superstring theory is,

$$S = \frac{1}{2k^2} \int d^{10}x \sqrt{-g}\, e^{-2\phi} \left[\left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) - \frac{\alpha'}{4} e^{-\phi}\, Tr\left(F_{\mu\nu} F^{\mu\nu} \right) \right] \quad (9.4)$$

The antisymmetric tensor field $B_{\mu\nu}$ appears through its field strength,

$$H = dB - \frac{\alpha'}{4} (\omega_{YM} - \omega_L) \quad (9.5)$$

Where ω_{YM}, and ω_L represents the Yang-Mills and Lorentz Chern-Simons forms. This modification is essential for the Green-Schwarz anomaly cancellation mechanism which ensures quantum consistency and uniquely restricts the gauge groups to SO(32) and $E_8 \times E_8$.

M-theory is seen as in figure 9.2, is the non-perturbative framework that unifies all five consistent superstring theories and eleven-dimensional supergravity into a single higher-dimensional theory. It stands up from the realization that the different superstring theories, although formulated in ten dimensions and appearing distinct at weak coupling, are connected through dualities and represent different limits of one underlying theory. M-theory is naturally defined in eleven spacetime dimensions, where the extra dimension becomes visible at strong coupling and its low energy effective description is given by eleven-dimensional supergravity. Unlike string theory where one-dimensional strings are fundamental objects, M-theory contains higher-dimensional extended objects known as branes, in particular the M2-brane and M5-brane. These membranes couple to a three-form gauge field present in eleven-dimensional supergravity and arise as dynamical supersymmetric solutions of the theory. Upon compactification, M2-branes and M5-branes reduce to fundamental strings, D-branes, and NS-branes of ten-dimensional string theories, showing that strings themselves are emergent objects within a more general brane framework. The action of M-theory is given by eleven-dimensional supergravity plus brane source,

$$S_M = \frac{1}{2k_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right) - \frac{1}{12k_{11}^2} \int C_3 \wedge F_4 \wedge F_4 + S_{M2} + S_{M5} \quad (9.6)$$

This action unifies superstring theory and supergravity in a single framework. The first term is the Einstein-Hilbert action in eleven dimensions and describes gravity, while the second term governs the dynamics of a three-form gauge field whose field strength F_4 couples naturally to membranes. The Chern-Simons term is essential for anomaly cancellation and brane consistency. The additional terms S_{M2} and S_{M5} are the worldvolume actions of the M2-brane and M5-brane, which

represent the fundamental extended objects of M-theory. When this action is compactified on different internal manifolds, it reproduces the actions of all known superstring and ten-dimensional supergravity theories. Compactifications on a circle yields Type IIA superstring theory, with the string coupling identified with the radius of the eleventh dimension, while compactification on an interval leads to the heterotic $E_8 \times E_8$ theory with gauge fields localized on the boundaries. Through chains of T- and S- dualities, the remaining string theories are connected within the same moduli space, demonstrating that they are not independent theories but different perturbative expansions of M-theory. M-theory is currently the most comprehensive framework for unifying quantum gravity and gauge interactions, but it lacks a complete formulation, predictive uniqueness, and experimental verification and therefore cannot yet be regarded as the final theory of physics.
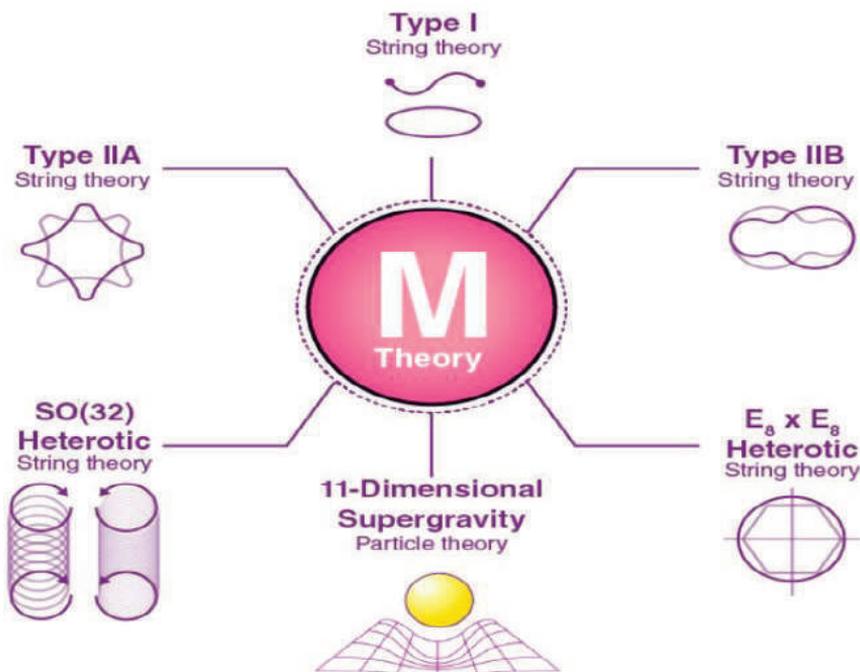

Figure 9.2: Representation of M-theory and their different types.

Chapter # 10

Theory of everything

The search for theory of everything is seen as in figure 10, is the effort to describe all of nature within single framework. This search began with Newton, whose laws of motion and gravitation unified terrestrial and celestial phenomena and established the idea that the universe is governed by universal laws. Despite its success, Newtonian mechanics treated space and time as absolute and offered no explanation for the nature of gravity, failing at relativistic speeds and microscopic scales. Einstein replaced this classical picture with relativity, unifying space and time and interpreting gravity as the curvature of spacetime. General relativity provided a deeper and fundamental description of gravity and successfully explained cosmic and strong field phenomena. However, it remained the classical theory incompatible with quantum principles and unable to describing physics at singularities or unifying gravity with other forces. Quantum mechanics introduced a radically different view of nature at small scales, revealing intrinsic uncertainty and wave particle duality. While extraordinary successful in explaining atomic and subatomic phenomena, it depends on a fixed background spacetime and excluded gravity, preventing it from serving as a complete description of the universe. The combination of quantum mechanics and special relativity led to quantum field theory and the standard model, which unifies electromagnetic, weak and strong interaction with remarkable precision. Nevertheless, this framework contains unexplained parameters, does not incorporate gravity, and fails to explain dark matter, dark energy and origin of spacetime. Attempt to quantize gravity with this approach breakdown, indicating the need for new theoretical foundation. String theory proposed such a foundation by replacing point particles with extended objects, naturally incorporating gravity and unifying all interactions with a single framework. It revealed deep dualities and required extra dimensions and supersymmetry for consistency. Yet string theory does not uniquely predict the structure of our universe and lacks experimental conformation preventing it from being a final theory of quantum gravity. M-theory emerges as a deeper framework unifying all string theories and supergravity, introducing higher dimensional objects and suggesting that spacetime and matter arise from a more fundamental structure. Although it represents it represents the most comprehensive unification attempt to date, its fundamental principles remain incomplete and it does not yet explain why our universe has its specific laws and constants.

The failure of each theory to achieve final unification shows that a true theory of everything must explain not only forces and particles but also the origin of spacetime, physical laws, and fundamental constants. When such a theory is found, it will address questions about the beginning of the universe, the nature of time, the quantum structure of gravity, the interior of blackholes, and the ultimate reason the universe exists. Dark matter will appear as a natural consequence of the theory's structure, while dark energy will be understood as an intrinsic property of the quantum vacuum, explaining cosmic acceleration and the smallness of vacuum energy. Cosmology will be placed on a first principles basis, clarifying the universe beginning, evolution and ultimate fate.

Information will be preserved in all physical processes, resolving black hole paradoxes and unifying gravity with quantum information. Such a theory will determine whether the universe is unique or one of many and will define the ultimate limits of scientific explanations. The path from Newton to M-theory thus represents not the end of physics, but the gradual approach toward a deeper understanding of reality.

Fig.10: The graphical representation of theory of everything where all laws of physics unify by a single theory of physics.

GLOSSARY

Celestial objects: Natural objects in space such as stars, planets, moons, asteroids and comets.

Equant: A point introduced in Ptolemy's model from which planetary motion appears uniform, even though it is not uniform around earth.

Rotational motion: Motion in which a body moves from one place to another without rotating.

Retrograde motion: The apparent backward motion of a planet in the sky relative to the stars for example observed Mars.

Cosmos: The universe regraded as an ordered and harmonies system.

Ptolemaic system: A geocentric cosmological model developed by Claudius Ptolemy in the 2nd century CE, placing a stationary Earth at the center of the universe.

Epicycle: A small circle whose center moves around the circumference of a larger circle.

Sunspots: Temporary dark, and relatively cool patches on the Sun's photosphere caused by intense, concentrated magnetic field lines that inhibit convection.

Inertia: The fundamental property of matter to resist changes in its velocity, including speed or direction of motion.

Uniform Acceleration: The rate of change of velocity at constant speed.

Elliptical Orbit: The path followed by a satellite that is shaped like an ellipse, characterized by two focal points with the Earth at one focus, and includes points of closest approach and farthest distance from earth.

Areal Velocity: The rate at which the radius vector (line from central point) to a moving particle sweeps out area expressed as area per unit time.

Semi-major axis: Half of the longest diameter of an ellipse, representing the distance from the center through a focus to the farthest edge.

Coulomb's Force: The force of attraction or repulsion between two charged particles.

Permittivity of free space: The capability of vacuum to permit electric charge or field.

Permeability of free space: Measurement of the ability of a material to support the formation of a magnetic field within the material.

Tensor: A mathematical object that generalizes scalars, vectors, and matrices to higher dimensions.

Gravitational lensing: The bending of light from a distant source around a massive foreground object.

Black hole: The region in space where an enormous amount of mass is packed into a tiny volume.

Worm hole: A hypothetical structure that connects two distinct regions in space.

White hole: The time reversal of black hole.

Singularity: A point in space where density and curvature reached to infinity inside black hole.

Geodesic: The generalization of a straight line to curved surfaces and representing the shortest possible path between two points.

Event horizon: The theoretical spherical boundary surrounding a black hole beyond which the gravitational pull is so strong that escape velocity exceeds the speed of light.

Photoelectric effect: When light strike the metal and electrons can be ejected from the surface of the metal.

Matter waves: The wave-like behavior of particles, described by quantum mechanics that reveals the dual nature of matter.

Time dilation: The perception of time varies depending on the relative motion of observers.

Length contraction: A relativistic phenomenon where an object length measured by an observer in motion relative to it, appears shorter along the direction of motion than its proper length.

Quantum fluctuations: Temporary, random changes in energy at the smallest scale.

Bosons: Fundamental particles that have spin in integer values (0,1,2 etc.).

Fermions: Fundamental particles with half integer spin (1/2, 3/2 etc.).

Lagrangian density: Kinetic energy minus potential energy.

Hamiltonian: The total energy of the physical system.

Spinor: The elements of a complex vector space that can be associated with Euclidean space.

Photon: The elementary field particle of electromagnetic interactions.

Gluon: The elementary field particle of strong nuclear interactions.

Gauge theory: Field theory in which the lagrangian and dynamics of the system itself, does not change under local transformations.

Feynman diagrams: Virtual representations of complex quantum field theory calculations, how subatomic particles interact, scatter, or annihilate.

Hyperspaces: Space with more than three dimensions.

Triads: Fundamental variables in loop quantum gravity representing the spatial geometry.

Diffeomorphism: it is an invertible function that maps one differentiable manifold to another.

Holonomy: A concept in differential geometry measuring how geometric data changes when parallel transported along a closed loop on a curved surface or manifold.

Flux: The amount of a vector field passing through a given surface representing the flow rate across that area.

Graviton: A hypothetical elementary particle that mediates the force of gravitational interaction.

Dilation: In string theory dilation is a scalar field, a fundamental massless particle that determines the strength of string interaction acting as a dynamic string coupling constant.

Ricci scalar: The simplest invariant measurement of curvature in Riemannian geometry.

R-R coupling: These topological couplings, often expressed through Chern-Simons terms on the world-volume, define the charge of the D-brane.

Supergravity: A quantum field theory that merges general relativity with supersymmetry providing a framework to describe gravity at the subatomic level.

Supersymmetry: It proposes that for every known particle, there exists a partner particle with different spin properties.

S-duality: A concept in physics that reveals an equivalence between two theories describing the same physical phenomena in different ways.

T-duality: A symmetry of the heterotic string to all orders in string perturbation theory.

U-duality: A symmetry of string theory which combining the T-duality with S-duality.

Membranes: Fundamental higher-dimensional extended objects that generalize 1D strings to 2D surfaces (membranes) and 5D objects within an 11-dimensional framework.

References

1. Hestenes D. New foundations for classical mechanics. Dordrecht: Springer Netherlands; 1999 Sep 30.
   DIO: https://doi.org/10.1007/0-306-47122-1_7
2. Benacquista MJ, Romano JD. Classical mechanics. Springer International Publishing; 2018 Feb 27.
   DOI: https://doi.org/10.1007/978-3-319-68780-3
3. Moritz H. Introduction to classical mechanics. InTheory of Satellite Geodesy and Gravity Field Determination 2005 Oct 20 (pp. 9-68). Berlin, Heidelberg: Springer Berlin Heidelberg.

DIO: https://doi.org/10.1007/BFb0010548

4. Magid, Leonard M., and Hermann A. Haus. "Electromagnetic Fields, Energy, and Waves." (1973): 57-59.
   DIO: https://doi.org/10.1063/1.3128054

5. Rachidi F, Tkachenko S. Electromagnetic field interaction with transmission lines: from classical theory to HF radiation effects. WIT press; 2008.
   DIO: https://doi.org10.1002/9781118057926

6. Soffel MH, Han WB. Applied general relativity. Astronomy and Astrophysics Library. Springer Nature Switzerland, Cham. 2019 Oct.

   DIO: https://doi.org/10.1007/978-3-030-19673-8

7. Khriplovich IB. General Relativity. New York, NY: Springer New York; 2005 Nov 16.
   DIO: https://doi.org/10.1007/0-387-27406-5_6

8. Moyal JE. Quantum mechanics as a statistical theory. In Mathematical Proceedings of the Cambridge Philosophical Society 1949 Jan (Vol. 45, No. 1, pp. 99-124). Cambridge University Press.

   DOI: https://doi.org/10.1017/S0305004100000487

9. McIntyre DH. Quantum mechanics. Cambridge University Press; 2022 Sep 15.

   DIO: https://doi.org/10.1007/3-540-28805-8_17

10. Wachter A. Relativistic quantum mechanics. Berlin: Springer; 2011.

    DIO: https://doi.org/10.1007/978-90-481-3645-2

11. Maiani L, Benhar O. Relativistic quantum mechanics: an introduction to relativistic quantum fields. CRC Press; 2024 Oct 16.

    DIO: https://doi.org/10.1201/9781003436263

12. Sterman, George, and Meinhard E. Mayer. "Introduction to quantum field theory." (1994): 94-94.
    DOI: https://doi.org/10.1063/1.2808706

13. Wipf, Andreas. "Statistical approach to quantum field theory: an introduction." (2013).

    DIO: https://doi.org/10.1007/978-3-642-33105-3

14. Rovelli C. Loop quantum gravity. Living reviews in relativity. 2008 Dec;11(1):5.
    DOI: https://doi.org/10.12942/lrr-2008-5

15. Ashtekar, Abhay, and Eugenio Bianchi. "A short review of loop quantum gravity." *Reports on Progress in Physics* 84, no. 4 (2021): 042001.
    DIO: https://doi.org/10.1088/1361-6633/abed9

16. Conlon J. Why string theory? CRC Press; 2016 Aug 19.
    DIO: https://doi.org/10.1201/9781315272368
17. Năstase H. Cosmology and string theory. Springer International Publishing; 2019 Apr 4.
    DOI: https://doi.org/10.1007/978-3-030-15077-8