

CHAPTER 11

MORE RECENT APPROXIMATE SYMMETRY THEORIES

In the previous chapter, three approximate symmetry theorems appearing in the literature has already been discussed. In this chapter, three additional approximate symmetry theories will be discussed which was proposed by Pakdemirli (2024b) recently. Some sample ODEs having exact solutions are used for the task of comparing the performances of the new methods. The symmetries as well as solutions are compared with each other. It is shown that Lie algebras may be extended, symmetries of the approximate methods may be retrieved from the exact symmetries or symmetries may be only a subgroup of the exact symmetries depending on the specific method and equation considered. With the aid of approximate symmetries, both approximate solutions and exact solutions can be retrieved.

11.1. DEFINITIONS OF APPROXIMATE SYMMETRY THEORIES

First, the definitions of extra three symmetries are given in this section. With reference to the three approximate symmetry theories given in the previous chapter, the new definitions are numbered as IV-V and VI.

Definition 11.1 (Approximate Symmetry Definition IV)

For the ordinary differential equation

$$F(x, y, y', y'', \dots, y^{(k)}, \varepsilon) = 0, \quad (11.1)$$

with ε being both the perturbation parameter and the Lie group of transformation parameter, the first order approximate symmetry corresponds to

$$F|_{\varepsilon=0} + \varepsilon XF|_{\varepsilon=0} = 0, \quad (11.2)$$

where

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial \varepsilon} + \eta^1 \frac{\partial}{\partial y_1} + \dots + \eta^k \frac{\partial}{\partial y_k}, \quad (11.3)$$

is the approximate symmetry generator extended to k 'th order. The group transformations and the extended infinitesimals are

$$x^* = x + \varepsilon \xi(x, y, \varepsilon), \quad (11.4)$$

$$y^* = y + \varepsilon \eta(x, y, \varepsilon), \quad (11.5)$$

$$y_1^* = y_1 + \varepsilon \eta^1(x, y, y_1, \varepsilon), \quad (11.6)$$

⋮

$$y_k^* = y_k + \varepsilon \eta^k(x, y, y_1, \dots, y_k, \varepsilon), \quad (11.7)$$

$$\mu^* = \varepsilon \mu \quad (11.8)$$

where

$$y_k = y^{(k)}, \quad \eta^k = \frac{D\eta^{k-1}}{Dx} - y_k \frac{D\xi}{Dx}, \quad \frac{D}{Dx} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \dots + y_{k+1} \frac{\partial}{\partial y_k} \quad (11.9)$$

The difference of this definition from the exact symmetries is that, the whole block (11.2) is used in calculating the symmetries whereas in exact symmetries, the two terms are separated. Another difference is that the Lie group of transformation parameter is different than the perturbation parameter in the case of exact symmetries whereas, in the above definition, they are the same.

Another definition of approximate symmetry may be as follows:

Definition 11.2 (Approximate Symmetry Definition V)

For the ordinary differential equation

$$F(x, y, y', y'', \dots, y^{(k)}, \varepsilon) = 0, \quad (11.10)$$

with ε being both the perturbation parameter and the Lie Group of transformation parameter, the first order approximate symmetry corresponds to

$$XF|_{\varepsilon=0} = 0 \quad \text{when} \quad F|_{\varepsilon=0} = 0, \quad (11.11)$$

where

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial \varepsilon} + \eta^1 \frac{\partial}{\partial y_1} + \dots + \eta^k \frac{\partial}{\partial y_k}, \quad (11.12)$$

is the approximate symmetry generator extended to k 'th order. The group transformations and the extended infinitesimals are

$$x^* = x + \varepsilon \xi(x, y), \quad (11.13)$$

$$y^* = y + \varepsilon \eta(x, y), \quad (11.14)$$

$$y_1^* = y_1 + \varepsilon \eta^1(x, y, y_1), \quad (11.15)$$

⋮

$$y_k^* = y_k + \varepsilon \eta^k(x, y, y_1, \dots, y_k), \quad (11.16)$$

$$\mu^* = \varepsilon \mu \quad (11.17)$$

where

$$y_k = y^{(k)}, \quad \eta^k = \frac{D\eta^{k-1}}{Dx} - y_k \frac{D\xi}{Dx}, \quad \frac{D}{Dx} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \dots + y_{k+1} \frac{\partial}{\partial y_k} \quad (11.18)$$

Comparing definition VI and V, one observes that the two terms of the block (11.2) in the former version is separated into two parts in the latter version. Another difference is that the infinitesimals $\xi(x, y)$ and $\eta(x, y)$ do not depend on the perturbation parameter, while there is the dependence of this parameter for Approximate Symmetry Method IV.

The differences of this new version from the exact symmetries are: 1) In the exact symmetries, perturbation parameter and Lie group of transformation parameter are different whereas they are the same. 2) In the exact case, the total differential equation is taken, i.e., $F = 0$, whereas in this approximate definition, the unperturbed equation $F|_{\varepsilon=0} = 0$ is taken to eliminate some of the higher order variables.

A modification of Approximate Symmetry Method IV may be as follows:

Definition 11.3 (Approximate Symmetry Definition VI)

For the ordinary differential equation

$$F(x, y, y', y'', \dots, y^{(k)}, \varepsilon) = 0, \quad (11.19)$$

with ε being the perturbation parameter and α being the Lie Group of transformation parameter, the first order approximate symmetry corresponds to

$$F|_{\alpha=0} + \alpha XF|_{\alpha=0} = 0, \quad (11.20)$$

where

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial y_1} + \dots + \eta^k \frac{\partial}{\partial y_k}, \quad (11.21)$$

is the approximate symmetry generator extended to k 'th order. The group transformations and the infinitesimals are

$$x^* = x + \alpha \xi(x, y, \varepsilon), \quad (11.22)$$

$$y^* = y + \alpha \eta(x, y, \varepsilon), \quad (11.23)$$

$$y_1^* = y_1 + \alpha \eta^1(x, y, y_1, \varepsilon) \quad (11.24)$$

⋮

$$y_k^* = y_k + \alpha \eta^k(x, y, y_1, \dots, y_k, \varepsilon), \quad (11.25)$$

where

$$y_k = y^{(k)}, \quad \eta^k = \frac{D\eta^{k-1}}{Dx} - y_k \frac{D\xi}{Dx}, \quad \frac{D}{Dx} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \dots + y_{k+1} \frac{\partial}{\partial y_k} \quad (11.26)$$

The difference of this definition with the exact symmetries is that in this definition, the whole block (11.20) is used whereas the two terms are separated in the exact case for the task of determining the infinitesimals. It turns out that this definition does not extend the Lie algebra of the exact symmetries which is our main goal in employing approximate theories. A subgroup of the Lie algebra of exact symmetries can be produced with this definition. The aim in including this new definition is therefore to outline the limitations of this definition and making comparisons with the previous approximate as well as exact symmetry definitions. Another reason to include the last theory is to show the importance of selecting the perturbation parameter as the Lie Group parameter as was done in definitions IV and V.

11.2. APPROXIMATE SYMMETRY CALCULATIONS

With the guidance of the definitions, approximate symmetries can be calculated. Several ordinary differential equations for first order, second and third orders are employed for comparison purposes. Skipping the details of the calculations, the exact and approximate infinitesimals are given in Table 11.1. Note that first

order solutions cannot be separated and solved in case of exact symmetries and the determining equations are given in unsolved form for them.

Table 11.1. Exact and Approximate Symmetries

Equation	Exact Symmetry	Approximate Symmetry IV	Approximate Symmetry V	Approximate Symmetry VI
$y' + \varepsilon y = 0$	Unsolvable $\eta_x + \varepsilon(\eta - y\eta_y + y\xi_x)$ $-\varepsilon^2\xi_y y^2 = 0$	$\xi = -\frac{\mu}{2}x^2 + a_1x + a_2$ $\eta = \left(-\mu x + a_1 - \frac{1}{\varepsilon}\right)y + b$	$\xi = \xi(x, y)$ $\eta = -\mu xy + a(y)$	$\xi = a + b e^{-\varepsilon x}$ $\eta = -(\varepsilon b e^{-\varepsilon x} + \frac{1}{\alpha})y + c e^{-\varepsilon x}$
$y' + e^{\varepsilon y} = 0$	Unsolvable $\eta_x - e^{\varepsilon y}(\eta_y - \xi_x - \varepsilon\eta)$ $-\xi_y e^{2\varepsilon y} = 0$	$\xi = -\frac{\mu}{2}x^2 + a_1x + a_2$ $\eta = \left(-\mu x + a_1 - \frac{1}{\varepsilon}\right)y - \frac{1}{\varepsilon}x + b_1$	$\xi = \xi(x, y)$ $\eta = -\xi + \mu \frac{y^2}{2} + a(x + y)$	$\xi = \frac{1}{\alpha}x + b$ $\eta = -\frac{1}{\varepsilon\alpha}$
$y'' + \varepsilon y'^2 = 0$	$\xi = (ax + b)e^{\varepsilon y} + cx^2 + dx + e$ $\eta = \left(f e^{-\varepsilon y} + \frac{c}{\varepsilon}\right)x + g e^{-\varepsilon y} + h + \frac{d}{\varepsilon} e^{\varepsilon y}$	$\xi = ax + b$ $\eta = \left(2a - \frac{1}{\varepsilon}\right)y + cx + d$	$\xi = (a_2x + a_3)y + c_1x^2 + b_1x + b_2$ $\eta = (2a_2 - \mu)\frac{y^2}{2} + (c_1x + c_2)y + d_1x + d_2$	$\xi = \frac{1}{2\alpha}x + a$ $\eta = b$
$y'' - 2\varepsilon yy' = 0$	$\xi = ax + b$ $\eta = -ay$	$\xi = a_2x + a_3$ $\eta = \left(2a_2 - \frac{1}{\varepsilon}\right)y + b_1x + b_2$	$\xi = \left(\frac{\mu}{3}x^2 + a_2x + a_3\right)y + c_1x^2 + b_1x + b_2$ $\eta = \left(\frac{2\mu}{3}x + a_2\right)y^2 + (c_1x + c_2)y + d_1x + d_2$	$\xi = ax + b$ $\eta = -ay$
$y'' - y + \varepsilon y^2 = 0$	$\xi = a$ $\eta = 0$	$\xi = a$ $\eta = -\frac{1}{\varepsilon}y + b_1e^x + b_2e^{-x}$	$\xi = (a_1e^x + a_2e^{-x})y + b_1 + b_2e^{2x} + b_3e^{-2x}$ $\eta = (a_1e^x - a_2e^{-x})y^2 + (c_1 + b_2e^{2x} - b_3e^{-2x})y + d_1e^x + d_2e^{-x}$	$\xi = 0$ $\eta = 0$
$y''' = \varepsilon f(y', y'')$	$\xi = a$ $\eta = b$	$\xi = a_1x + a_2$ $\eta = \left(3a_1 - \frac{1}{\varepsilon}\right)y + b_1x^2 + b_2x + b_3$	$\xi = a_1x^2 + a_2x + a_3$ $\eta = (2a_1x + a_2 + c)y + b_1x^2 + b_2x + b_3$	$\xi = 0$ $\eta = 0$

Inspecting the above table, some conclusions can be drawn from the exact and approximate symmetries.

For first order equations;

- The exact infinitesimals cannot be determined in their most general forms unless some simplifying assumptions are made to solve the determining equation blocks. As pointed in the earlier chapters, the problem stems from the inseparable single equation block which is not adequate to determine two unknowns.
- In contrast, for the approximate symmetry theories, the determining block is separable and the approximate infinitesimals can be determined in their most general form.
- The highest number of symmetries can be obtained using Approximate Symmetry Theory V.

For the higher order equations;

- Regarding the second order equation $y'' + \varepsilon y'^2 = 0$, the exact and Approximate Symmetry Theory V both produced 8-finite parameter Lie Group of transformations. In contrast, Approximate Symmetry Theory IV produced 4-finite parameter Lie Group of transformations and Approximate Symmetry Theory VI produced 2-finite parameter Lie Group of transformations.
- For the equation $y'' + \varepsilon y'^2 = 0$, the Approximate Symmetry Theory V results are retrievable from the exact ones by expanding in a Taylor series and keeping terms up to $O(\varepsilon)$.
- For the remaining 3 equations at the bottom, Approximate Symmetries IV and V are more in number than the exact ones. For the equation $y'' - y + \varepsilon y^2 = 0$, the exact symmetries are one parameter while the Approximate Symmetry IV and V are 3 and 8 parameters respectively.
- One general rule is that the Approximate Symmetry VI is either equal or subalgebra of the exact symmetries.
- Generally speaking, the richest symmetries can be obtained by Approximate Symmetry Theory V compared to the other approximate symmetry theories.

11.3. EXACT AND APPROXIMATE SOLUTIONS

Symmetries will be employed to produce group invariant solutions in this section. The determining equations for group invariant solutions are

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}. \quad (11.27)$$

Four of the problems are selected from Table 11.1 to construct group invariant solutions. Results are shown in Table 11.2. The exact and first order approximate solutions are given in the second column. The third column is devoted to the specific exact symmetry to retrieve the solutions. The last right three columns are for the approximate infinitesimals corresponding to different theories to retrieve such solutions.

Table 11.2. Group Invariant Solutions

Equation	Exact and Approximate Solutions	Exact Symmetry	Approximate Symmetry IV	Approximate Symmetry V	Approximate Symmetry VI
$y' + \varepsilon y = 0$ $y(0) = 1$	$y_e = e^{-\varepsilon x}$	Retrievable	$\xi = a_2$ $\eta = -\frac{1}{\varepsilon}y$	$\xi = b$ $\eta = y$	$\xi = be^{-\varepsilon x}$ $\eta = -\varepsilon be^{-\varepsilon x}y$
	$y_a = 1 - \varepsilon x$	Not directly retrievable	$\xi = a_2$ $\eta = b$	$\xi = 1$ $\eta = a$	Not directly retrievable
$y' + e^{\varepsilon y} = 0$ $y(0) = 0$	$y_e = -\frac{1}{\varepsilon} \ln(1 + \varepsilon x)$	Not directly retrievable	Not directly retrievable	Not directly retrievable	$\xi = \frac{1}{\alpha}x + b$ $\eta = -\frac{1}{\varepsilon\alpha}$
	$y_a = -x + \varepsilon \frac{x^2}{2}$	Not directly retrievable	$\xi = a_2$ $\eta = -\frac{1}{\varepsilon}x + b_1$	Not directly retrievable	Not directly retrievable
$y'' + \varepsilon y'^2 = 0$ $y(0) = 0$ $y'(0) = 1$	$y_e = \frac{1}{\varepsilon} \ln(1 + \varepsilon x)$	$\xi = dx + e$ $\eta = h$	$\xi = ax + b$ $\eta = d$	$\xi = b_1x + b_2$ $\eta = d_2$	$\xi = \frac{1}{2\alpha}x + a$ $\eta = b$
	$y_a = x - \varepsilon \frac{x^2}{2}$	$\xi = e$ $\eta = \frac{e}{\varepsilon}x + h$	$\xi = b$ $\eta = cx + d$	$\xi = b_2$ $\eta = d_1x + d_2$	Not directly retrievable
$y'' - 2\varepsilon yy' = 0$ $y(0) = 1$ $y'(0) = \varepsilon$	$y_e = \frac{1}{1 - \varepsilon x}$	$\xi = ax + b$ $\eta = -ay$	$\xi = a_2x + a_3$ $\eta = \left(2a_2 - \frac{1}{\varepsilon}\right)y$	$\xi = b_1x + b_2$ $\eta = c_2y$	$\xi = ax + b$ $\eta = -\alpha y$
	$y_a = 1 + \varepsilon x$	Not directly retrievable	$\xi = a_3$ $\eta = b_2$	$\xi = b_2$ $\eta = d_2$	Not directly retrievable

From the table, it is evident that Approximate Symmetry Theories IV and V are better in constructing the solutions. Approximate Symmetry VI failed to produce approximate solutions since it is only a subalgebra of the exact symmetries.

In the majority of the cases, the approximate infinitesimals yield the exact solutions also. The reason might be that the dependent variable is not expanded in a perturbation series. Similar feature is also observed for Approximate Symmetry Theory I discussed in the previous chapter. This choice of not expanding the dependent variable may be questioned from the perturbation theory point of view (Pakdemirli et al., 2004). The main difference however with the Approximate Symmetry Theory I and the ones presented here (Approximate Symmetry Theories IV-VI) is that in the former, an approximate generator is calculated while there is no need for such a calculation in the newly proposed ones.

11.4. CONCLUSIONS

Based on the work presented in this chapter and in the previous chapter, some general comments can be made:

- If the main aim is to retrieve only the approximate solutions, Approximate Symmetry Theory II or Approximate Symmetry Theory III with less

algebra is recommended, which are more consistent with the essentials of perturbation theory leading directly to the approximate solutions.

- If the main aim is to retrieve both the approximate and exact solutions, Approximate Symmetry Theories I, IV and V can be used.
- Among the three approximate symmetry methods presented in this chapter, Approximate Symmetry Theory V is mostly recommended for second and higher order equations due to its higher number of symmetries.
- Approximate Symmetry Theory IV has an advantage for first order ordinary differential equations, that is, it leads to simpler and solvable infinitesimals compared to exact and Approximate Symmetry Theory V cases.
- Approximate Symmetry Theory VI produces only an equivalent or subgroup of the exact symmetries which corresponds to developing group invariant solutions. The theory is presented only for comparison purposes and not recommended for practical usage.

11.5. EXERCISES

E.11.1. Calculate all the exact and approximate infinitesimals given in Table 11.1.

E.11.2. Calculate the exact and approximate solutions from the given infinitesimals listed in Table 11.2.

FURTHER TOPICS

In this book, excluding the last two chapters, only the so-called Lie-point symmetries are considered. There exist many generalizations of Lie-point symmetries as well as alternative methods in determining the symmetries which is hard to discuss in the context of an introductory book.

In the point symmetries, the infinitesimals are assumed to depend on the independent and dependent variables, i.e. $\xi_i = \xi_i(x_i, u^\alpha)$, $\eta^\alpha = \eta^\alpha(x_i, u^\alpha)$, $i = 1, 2, \dots, n$, $\alpha = 1, 2, \dots, m$. The first logical generalization is the contact symmetries in which the infinitesimals depend on the first derivatives also; $\xi_i(x_i, u^\alpha, u_i^\alpha)$, $\eta^\alpha = \eta^\alpha(x_i, u^\alpha, u_i^\alpha)$ (Bluman and Kumei, 1989; Olver, 1986; Stephani, 1989; Ibragimov, 1999). If there is more than one dependent variable, i.e. $\alpha \geq 2$, then it is shown that contact symmetries reduce to those of Lie-point symmetries. Hence, contact symmetries are of use in case of one dependent variable only. A further generalization is the Lie-Backlund symmetries in which case the infinitesimals depend on arbitrary orders of derivatives $\xi_i(x_i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, u_{ijk}^\alpha, \dots)$, $\eta^\alpha = \eta^\alpha(x_i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, u_{ijk}^\alpha, \dots)$ (Bluman and Kumei, 1989; Olver, 1986; Stephani, 1989; Ibragimov, 1999). Lie-Backlund symmetries are especially useful in linking conservation laws and Lagrangians of the system. Noether's theorem employs the symmetries in constructing the conservation laws. When higher order derivatives are included, the determining equations for infinitesimals become more and more complex and separation with respect to higher order variables become increasingly difficult as the dependence of the infinitesimals on the higher orders of derivatives are increased. Subgroups of such symmetries can be determined by employing some simplifying assumptions or truncation of the infinite order of derivatives to finite orders.

Non-classical symmetry is another type of extension for Lie-point symmetries first proposed by Bluman and Cole (1969). By using some side conditions, similarity solutions that cannot be retrieved by Lie-point symmetries can be obtained with the non-classical symmetries. The example of well-known heat conduction equation is extensively analyzed to display new similarity solutions (Bluman and Cole, 1969).

Symmetries can also be employed in mapping nonlinear partial differential equations to linear ones or linear partial differential systems with variable

coefficients to linear partial differential systems with constant coefficients. The algorithms for such transformations were outlined by Bluman and Kumei (1989). For a nonlinear PDE that can be transformed into a linear PDE, the original nonlinear equation should have infinite parameter Lie group of transformations, a characteristic feature of linear PDEs.

If the infinitesimals depend on integrals of dependent variables, then one can speak of non-local symmetries or potential symmetries. Differentiation or integration of a differential equation may lead to such symmetries. Bluman and Kumei (1989) and Baumann (2000) outlined the calculation algorithms for such symmetries. As an example of non-local symmetries, consider the constant acceleration curve equation

$$(1 + y'^2)y''' - (3y' + 2\varepsilon)y''^2 = 0 ,$$

which has local symmetries (Pakdemirli, 2023b)

$$\xi = ay + bx + c, \eta = -ax + by + d .$$

However, the order of the equation can be reduced by one by defining a new variable $p = y'$,

$$(1 + p^2)p'' - (3p + 2\varepsilon)p'^2 = 0 .$$

The symmetries of this reduced system contains integrals of the dependent variable

$$\xi = k_1x^2 + c(p)x + d(p), \eta = a(p)x + b(p) ,$$

where

$$a(p) = e^{2\varepsilon \tan^{-1}p} (1 + p^2)^{3/2} \left(k_1 \int \frac{e^{-2\varepsilon \tan^{-1}p}}{(1+p^2)^{3/2}} dp + k_2 \right) ,$$

$$c(p) = k_3 \int \frac{e^{-2\varepsilon \tan^{-1}p}}{(1+p^2)^{3/2}} dp + k_4 ,$$

$$d(p) = k_5 \int \frac{e^{-2\varepsilon \tan^{-1}p}}{(1+p^2)^{3/2}} dp + k_6 ,$$

and $b(p)$ satisfies

$$(1 + p^2)b'' - (3p + 2\varepsilon)b' + \frac{3p^2 - 3 + 4\varepsilon p}{1 + p^2}b = 2k_3 \frac{e^{-2\varepsilon \tan^{-1} p}}{(1 + p^2)^{1/2}}.$$

It is thereby observed that while the symmetries of the original system are local and simple, the symmetries of the reduced system are non-local and much more involved.

In Chapter 8, for equations having arbitrary parameters/functions other than the dependent variables, group classification problems with respect to these parameters/functions were treated. An alternative approach to this direct group classification method is to use the so-called equivalence transformations (Ibragimov, 1995). In the equivalence transformations, such parameters/functions are treated similar to the dependent variables and infinitesimal generators contain their derivatives as separate differential terms. A comparison of the classical approach and the equivalence transformation approach was given for the group classification problem of a non-Newtonian fluid flow (Yürüsöy and Pakdemirli, 1999b). It is concluded that for this specific problem, both approaches yield identical results. However, it may happen that a subgroup of the whole classification may be retrieved by the equivalence transformation, which is the reason of calling this alternative approach as the preliminary group classification.

Exterior differential form approach is an alternative approach to the classical Lie group theory to determine the symmetry generators. The method is proposed for the first time in a seminal paper by Harrison and Estabrook (1971). The theory is discussed in detail by Şuhubi (2010). A solution algorithm was outlined for solving general balance equations by employing the exterior calculus approach (Şuhubi, 1991) and this general formalism was adapted to boundary layer equations of second grade fluids (Pakdemirli and Şuhubi, 1992) and to radial motions of compressible heterogeneous hyperelastic spheres and cylinders (Şuhubi, 1994). Equivalence transformation approach has also been augmented into the exterior differential form method (Şuhubi 1998, Pakdemirli and Yürüsöy, 1999).

As a final topic, weak symmetries are the symmetries which are not admitted by the equation. The similarity variables do not properly transform the equation such that the transformed equation cannot be solely expressed in terms of these

new variables. However, the transformed equations may still possess a solution. As an example, consider the equation

$$u_t = uu_x + u_{xx}$$

which admits the scaling symmetry generator

$$X = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}.$$

Definitely, the generator

$$X = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}.$$

does not leave the equation invariant. Defining the similarity variables corresponding to this generator

$$\xi = \frac{x}{t}, \quad u = f(\xi),$$

the equation transforms into

$$f'' + tf'(f + \xi) = 0.$$

The equation did not transform properly, a coefficient of t remained uneliminated since $X = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}$ is not a symmetry of the equation. A solution can still be found by separating the above ODE

$$f'' = 0, \quad f'(f + \xi) = 0,$$

which definitely has a non-trivial joint solution $f = -\xi$ or

$$u = -\frac{x}{t}.$$

One can claim that $X = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}$ is a weak symmetry of $u_t = uu_x + u_{xx}$ and $u = -\frac{x}{t}$ is a weak solution corresponding to this symmetry. Systematic ways of determining such weak symmetries do not exist and the procedure is merely trial and error.

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