# CHAPTER 10

# APPROXIMATE SYMMETRY THEORIES

Although Lie group theory is a systematic and unified way of finding exact analytical solutions of differential equations, exact analytical solutions are rare for especially nonlinear and variable coefficient equations. If the equation does not admit an exact analytical solution, then no matter what method is employed, a solution cannot be expressed in the form of known functions. Lie group theory is not an exception, that is, if there is no analytical solution, then it cannot be deduced by Lie group theory also.

If analytical solutions are not available, the next best choice is to search for approximate analytical solutions before resorting to purely numerical techniques. One widely used and well established approximate analytical solution technique is the perturbation technique applied successfully for over a century to differential equations (Nayfeh, 1973, 1981, 1985; Murdock, 1999; Hinch, 1991; Shivamoggi, 2003; Aziz and Na, 1984; Kevorkian and Cole, 1981; Van Dyke, 1975). The idea is then to unite the potentials of perturbation methods and symmetry methods to develop new methods for finding approximate solutions systematically.

To achieve this task, the so called approximate symmetry methods were proposed in the literature. The first method proposed by Baikov et al. (1989), Ibragimov (1996) is the most common method used in the literature. The method relies on the approximate expansion of the generator rather than the approximate expansion of the dependent variable as in the case of a usual perturbation expansion. The second approximate symmetry method is due to Fushchich and Shtelen (1989) and later employed by Euler et al. (1992, 1994) and Euler and Euler (1994). In this method, the perturbation expansion of the dependent variable is substituted as an initial step and equations at each order are separated with respect to orders of the perturbation parameter as is done in a usual perturbation analysis. The outcoming equations are assumed to be coupled with respect to dependent variables at each order of approximation. The approximate symmetry is then defined to be the exact symmetries of the outcoming coupled equations. Since the dependent variables increase with the terms in the expansions, the algebra of determining symmetries increases much in this method. Most of the physical problems however contain an unperturbed linear part and a perturbed nonlinear part. In such cases, since the equations are solved in hierarchy, in fact, the equations are uncoupled equations. The system can be considered as a single linear equation with non-homogenous known parts coming from the previous orders of calculations but vary at each order of approximations. Requiring the non-homogenous part to be an arbitrary function of the independent variables, the approximate symmetry is then defined as the exact symmetry of the non-homogenous linear equation in the third method proposed by Pakdemirli et al. (2004) and Dolapci and Pakdemirli (2004). The third method and the second method are more close to each other than the first method in the sense that approximate symmetries are defined as the exact symmetries of the equations arising after separation of the equations with respect to orders. The third method is effective especially in reducing the computational algebra of calculating symmetries compared to the second method. In the next chapter, the recently proposed additional three approximate symmetry theories will also be discussed.

## 10.1. APPROXIMATE SYMMETRY GENERATORS

In this section, approximate symmetry generators corresponding to Methods I, II and III as well as exact symmetries will be presented for the potential Burgers equation and the symmetries will be contrasted with each other. The potential Burges equation is

$$u_t = u_{xx} + \varepsilon u_x^2 \, , \tag{10.1}$$

where $u = u(x,t)$ and $\varepsilon$ is the perturbation parameter. First, the exact symmetries and then the symmetries corresponding to each method are presented.

***Exact Symmetries***

The infinitesimal generator for the problem is

$$X = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \, . \tag{10.2}$$

Using the standard method outlined in Chapter 6, the infinitesimals are

$$\xi_1 = 4axt + bx + 2ct + d \, , \tag{10.3}$$

$$\xi_2 = 4at^2 + 2bt + e \, , \tag{10.4}$$

$$\eta = \frac{1}{\varepsilon}\theta(x,t)e^{-\varepsilon u} - \frac{a}{\varepsilon}x^2 - \frac{2a}{\varepsilon}t - \frac{c}{\varepsilon}x + f \, , \tag{10.5}$$

where

$$\theta_t = \theta_{xx} \, . \tag{10.6}$$

The exact symmetries consist of 6 finite parameter Lie group of transformations with one infinite parameter Lie group of transformation. The unperturbed equation, i.e., $\varepsilon = 0$, namely the heat equation has the symmetries

$$\xi_1 = 4axt + bx + 2ct + d \, , \tag{10.7}$$

$$\xi_2 = 4at^2 + 2bt + e \, , \tag{10.8}$$

$$\eta = -a(x^2 + 2t)u - cxu + fu + h(x,t) \, , \tag{10.9}$$

where

$$h_t = h_{xx} \, . \tag{10.10}$$

The unperturbed equation and the perturbed equation has the same number of symmetries albeit different, which is a rare case. Usually the symmetries of the perturbed equations are less than the unperturbed equations due to the inclusion of additional terms.

***Approximate Symmetry Method I***

In this method, the perturbed equation is expressed as

$$F(x,t,u,u_t,u_x,u_{xx};\varepsilon) = F_0 + \varepsilon F_1 = 0 \, . \tag{10.11}$$

The generator is also approximated

$$X = X_0 + \varepsilon X_1 \, . \tag{10.12}$$

Application of the generator to the perturbed equation yields the algorithm (Ibragimov, 1996)

*i*) Find the exact symmetry $X_0$ of the unperturbed equation

$$X_0 F_0 |_{F_0=0} \, . \tag{10.13}$$

*ii*) Calculate the auxiliary function

$$H \approx \frac{1}{\varepsilon} X_0 (F_0 + \varepsilon F_1)|_{F_0+\varepsilon F_1=0} \,. \tag{10.14}$$

*iii*) Find the first order deformations $X_1$

$$X_1 F_0|_{F_0=0} + H = 0 \,. \tag{10.15}$$

Skipping the details, the final results for Method I for the potential Burgers equation is

$$\xi_1 = 4axt + bx + 2ct + d + \varepsilon(4Axt + Bx + 2Ct + D) \,, \tag{10.16}$$

$$\xi_2 = 4at^2 + 2bt + e + \varepsilon(4At^2 + 2Bt + E) \,, \tag{10.17}$$

$$\eta = -a(x^2 + 2t)u - cxu + fu + h(x,t)$$

$$+\varepsilon[(a(x^2 + 2t) + cx - f)\frac{u^2}{2} - A(x^2 + 2t)u$$

$$-Cxu + Fu - uh(x,t) + g(x,t)] \,, \tag{10.18}$$

where

$$h_t = h_{xx} \,, \quad g_t = g_{xx} \,. \tag{10.19}$$

At first glance, one sees that there are many repetitions of the symmetries at the first order. These repetitions of symmetries can be considered as trivial symmetries since with new definition of parameters

$$\hat{a} = a + \varepsilon A, \;\; \hat{b} = b + \varepsilon B, \hat{c} = c + \varepsilon C, \;\; \hat{d} = d + \varepsilon D \,,$$

$$\hat{e} = e + \varepsilon E, \;\; \hat{f} = f + \varepsilon F, \hat{h}(x,t) = h(x,t) + \varepsilon g(x,t) \,, \tag{10.20}$$

they can be augmented into the first order infinitesimals

$$\xi_1 = 4\hat{a}xt + \hat{b}x + 2\hat{c}t + \hat{d} \,, \tag{10.21}$$

$$\xi_2 = 4\hat{a}t^2 + 2\hat{b}t + \hat{e} \,, \tag{10.22}$$

$$\eta = -\hat{a}(x^2 + 2t)u - \hat{c}xu + \hat{f}u + \hat{h}(x,t)$$

$$+\varepsilon[(a(x^2 + 2t) + cx - f)\frac{u^2}{2} - u\hat{h}(x,t)] \,. \tag{10.23}$$

$\xi_1$ and $\xi_2$ in exact and approximate cases look similar now. The relationship between the approximate $\eta$ and exact $\eta$ can be exploited by expanding the exponential term in (10.5) in a Taylor series up to second order and choosing $f = f/\varepsilon$,

$$\eta \cong \frac{1}{\varepsilon}[\theta(x,t) - ax^2 - 2at - cx + f] - u\theta(x,t) + \varepsilon\frac{u^2}{2}\theta(x,t)\ . \tag{10.24}$$

Without loss of generality, one may define $\theta(x,t) = ax^2 + 2at + cx - f + \varepsilon h(x,t)$ which when inserted into above yields

$$\eta \cong -a(x^2 + 2t)u - cxu + fu + h(x,t)$$

$$+\varepsilon[(a(x^2 + 2t) + cx - f)\frac{u^2}{2} - uh(x,t)]\ . \tag{10.25}$$

This approximate $\eta$ obtained from the exact one is identical to (10.23). Obtaining approximate symmetries from approximations of the exact generator worked for this problem. But for many of the problems, the $\varepsilon$ dependence of the exact symmetries does not appear and a transformation from exact to approximate is impossible. This has been outlined for the creeping flow equations of a non-Newtonian fluid (Pakdemirli et al., 2004). In summary, by Method I, many trivial symmetries as well as nontrivial symmetries appear as deformations of the generator. The generator rather than the dependent variable is expanded in a perturbation series.

***Approximate Symmetry Method II***

In this method, the perturbation expansion is substituted into the equation as an initial step (Fushchich and Shtelen, 1989; Euler et al., 1992, 1994; Euler and Euler, 1994). The resulting equations are assumed to be coupled and the approximate symmetry of the original equation would be the exact symmetry of the resulting separated equations. The approach is consistent with the perturbation theory as well as with the Lie group theory. Regarding the potential Burgers equation, the first order perturbation expansion

$$u = u_0 + \varepsilon u_1 + \cdots\ , \tag{10.26}$$

is substituted into (10.1) and separated with respect to orders

$$O(1):\ \frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} = 0\ , \tag{10.27}$$

$$O(\varepsilon): \ \frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = \left(\frac{\partial u_0}{\partial x}\right)^2 \ . \tag{10.28}$$

The infinitesimal generator is

$$X = \xi_1(x,t,u_0,u_1)\frac{\partial}{\partial x} + \xi_2(x,t,u_0,u_1)\frac{\partial}{\partial t}$$
$$+\eta^1(x,t,u_0,u_1)\frac{\partial}{\partial u_0} + \eta^2(x,t,u_0,u_1)\frac{\partial}{\partial u_1} \quad . \tag{10.29}$$

The straightforward calculations yield

$$\xi_1 = 4axt + bx + 2ct + d \ , \tag{10.30}$$

$$\xi_2 = 4at^2 + 2bt + e \ , \tag{10.31}$$

$$\eta^1 = -ax^2u_0 - cu_0x - 2au_0t - 2fu_0 + 2gu_1 + gu_0^2 + \alpha(x,t) \ , \tag{10.32}$$

$$\eta^2 = -ax^2u_1 - cu_1x - 2gu_0u_1 - 2au_1t - 2(f+j)u_1 + \tfrac{1}{2}ax^2u_0^2$$
$$+\tfrac{1}{2}cxu_0^2 - gu_0^3 + (f-j)u_0^2 + au_0^2t - \alpha(x,t)u_0 + \beta(x,t) \ , \tag{10.33}$$

$$\alpha_t = \alpha_{xx} \ , \quad \beta_t = \beta_{xx} \ . \tag{10.34}$$

Since there are two dependent variables, the algebra calculated is more involved compared to a single dependent variable. If more correction terms are included in the perturbation expansion, the algebra would increase enormously. The basic assumption is that one is trying to solve both orders under the coupled equation assumption which is not the case usually. Indeed, $u_0$ is uncoupled and can be solved first from (10.27). The second equation is then a similar non-homogenous linear equation that can be solved without repeating the symmetry calculations.

***Approximate Symmetry Method III***

The method can be considered as a modification of the previous method. The coupled equations assumption is removed. Following Pakdemirli et al. (2004) and Dolapci and Pakdemirli (2004), the method is outlined in a more general form first. Consider the nonlinear equation

$$\mathcal{L}(u) + \varepsilon N(u) = 0 \ , \tag{10.35}$$

where $\mathcal{L}$ and $N$ are the arbitrary linear and nonlinear operators respectively. The dependent variable is a function of independent variables $x_i, i = 1,2, \dots n$. The perturbation expansion

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots , \tag{10.36}$$

is substituted and separated with respect to orders

$$\mathcal{L}(u_0) = 0 = h_0(x_1, x_2, \dots , x_n) , \tag{10.37}$$

$$\mathcal{L}(u_1) = -N(u_0) = h_1(x_1, x_2, \dots , x_n) , \tag{10.38}$$

$$\mathcal{L}(u_2) = -N(u_0, u_1) = h_2(x_1, x_2, \dots , x_n) . \tag{10.39}$$

Indeed all equations are now linear with the right hand sides known functions determined sequentially starting from the first equation. The approximate symmetry is then defined as follows

**Definition 10.1.** The approximate symmetry of the nonlinear equation (10.35) is the exact symmetry of the linear non-homogenous equation

$$\mathcal{L}(u) = h(x_1, x_2, \dots , x_n) , \tag{10.40}$$

with $h$ being an arbitrary function of the independent variables

Since $h_i$ are indeed known functions, substituting them to the general symmetry equations of (10.40), the specific symmetries at each level of approximation can be determined. Returning back to the potential Burgers equation, the expansion

$$u = u_0 + \varepsilon u_1 + \cdots , \tag{10.41}$$

produces the separated equations

$$O(1): \ \frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} = 0 , \tag{10.42}$$

$$O(\varepsilon): \ \frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = \left(\frac{\partial u_0}{\partial x}\right)^2 . \tag{10.43}$$

At order 1, $h = 0$ and at the next order $h(x,t) = \left(\frac{\partial u_0}{\partial x}\right)^2$, a known function determined from the previous level of approximation. In this method, therefore symmetries of the below simple equation has to be calculated

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = h(x,t) \,. \tag{10.44}$$

The infinitesimal generator is

$$X = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \,. \tag{10.45}$$

The straightforward calculations yield

$$\xi_1 = 4axt + bx + 2ct + d \,, \tag{10.46}$$

$$\xi_2 = 4at^2 + 2bt + e \,, \tag{10.47}$$

$$\eta = -a(x^2 + 2t)u - cxu + fu + \alpha(x,t) \,, \tag{10.48}$$

$$\alpha_t - \alpha_{xx} - h[ax^2 + cx + 10at - f + 2b]$$
$$= [4axt + bx + 2ct + d]h_x + [4at^2 + 2bt + e]h_t \,. \tag{10.49}$$

At the first level, insert $h = 0$ into (10.49) which yields the symmetries of the unperturbed equation. At the subsequent levels, insert the calculated specific $h$ values into (10.49) to determine the symmetries at that level. No matter what the number of correction terms is, one always deals with a single equation reducing the algebra extensively corresponding to the coupled equations assumption of Method II.

## 10.2. APPROXIMATE SOLUTIONS

Approximate solutions corresponding to each method will be derived in this section for two of the symmetries.

***Parameter b***

*i*) *Exact solution*

First the exact solution will be derived from the exact symmetries. Selecting parameter *b* only, from (10.3)-(10.5)

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{0} \,, \tag{10.50}$$

which produces the similarity variables

$$\xi = \frac{x}{\sqrt{t}} \,, \quad u = f(\xi) \,. \tag{10.51}$$

Substituting into (10.1)

$$f'' + \frac{1}{2}\xi f' + \varepsilon f'^2 = 0 \, , \tag{10.52}$$

and solving yields

$$f = \int_0^{\xi} \frac{\exp(-\zeta^2/4)}{\varepsilon \int_0^{\zeta} \exp(-\eta^2/4) d\eta + c_1^*} d\zeta + c_2 \, , \tag{10.53}$$

where $c_1^*$ and $c_2$ are some constants. An approximate solution can be derived from the above exact solution by expanding the denominator and keeping terms of order $\varepsilon$

$$u = f = c_1 \int_0^{\xi} \exp(-\zeta^2/4)\, d\zeta + c_2$$

$$-\varepsilon c_1^2 \int_0^{\xi} \left[\exp(-\zeta^2/4) \int_0^{\zeta} \exp(-\eta^2/4) d\eta\right] d\zeta \, , \tag{10.54}$$

where $c_1 = 1/c_1^*$.

*ii*) *Method I*

To calculate the approximate solutions for Method I, use the symmetries (10.16)-(10.18). Parameters $b$ and $B$ are selected. However, only $b$ can also be considered. This is the reason of calling such symmetries as trivial since they do not provide additional information.

$$\frac{dx}{(b+\varepsilon B)x} = \frac{dt}{2(b+\varepsilon B)t} = \frac{du}{0} \, , \tag{10.55}$$

which produces the exact solution (10.53) containing not only the O($\varepsilon$) terms but also higher order terms.

*iii*) *Method II*

To calculate the approximate solutions for Method II, use the symmetries (10.30)-(10.33). Parameter $b$ yields

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du_0}{0} = \frac{du_1}{0} \, , \tag{10.56}$$

with similarity variables

$$\xi = \frac{x}{\sqrt{t}} \, , \quad u_0 = f(\xi) \, , \quad u_1 = g(\xi) \, . \tag{10.57}$$

Substituting into (10.27) and (10.28)

$$f'' + \frac{1}{2}\xi f' = 0 \,, \tag{10.58}$$

$$g'' + \frac{1}{2}\xi g' + f'^2 = 0 \,, \tag{10.59}$$

and solving yields

$$f = c_1 \int_0^{\xi} \exp(-\zeta^2/4)\, d\zeta + c_2 \,, \tag{10.60}$$

$$g = -c_1^2 \int_0^{\xi} \left[\exp(-\zeta^2/4) \int_0^{\zeta} \exp(-\eta^2/4) d\eta\right] d\zeta$$

$$+c_3 \int_0^{\xi} \exp(-\zeta^2/4)\, d\zeta + c_4 \,. \tag{10.61}$$

Inserting the results into (10.26), redefining the constants, the approximate solution is

$$u = c_1 \int_0^{\xi} \exp(-\zeta^2/4)\, d\zeta + c_2$$

$$-\varepsilon c_1^2 \int_0^{\xi} \left[\exp(-\zeta^2/4) \int_0^{\zeta} \exp(-\eta^2/4) d\eta\right] d\zeta \,, \tag{10.62}$$

which is the same approximate solution obtained by approximating the exact solution.

*iv) Method III*

To calculate the approximate solutions for Method III, use the symmetries (10.46)-(10.48). Parameter $b$ with $h = 0$ yields the first order solution

$$u_0 = c_1 \int_0^{\xi} \exp(-\zeta^2/4)\, d\zeta + c_2 \,. \tag{10.63}$$

Now, $h(x,t)$ for the next order is

$$h = \left(\frac{\partial u_0}{\partial x}\right)^2 = \frac{c_1^2}{t} exp(-\xi^2/2) \,. \tag{10.64}$$

Taking all parameters except $b$ zero in (10.46)-(10.48), it is observed that (10.49) is also satisfied for $h$ given in (10.64), and hence

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du_1}{0} \,, \tag{10.65}$$

which produces the similarity variables

$$\xi = \frac{x}{\sqrt{t}}, \quad u_1 = g(\xi) \,. \tag{10.66}$$

Substituting the above variables into (10.43) and solving yields

$$u_1 = g = -c_1^2 \int_0^{\xi} \left[\exp(-\zeta^2/4) \int_0^{\zeta} \exp(-\eta^2/4) d\eta\right] d\zeta$$

$$+c_3 \int_0^{\xi} \exp(-\zeta^2/4)\, d\zeta + c_4 \,. \tag{10.67}$$

Finally, inserting the results into (10.41), redefining the constants, the approximate solution is

$$u = c_1 \int_0^{\xi} \exp(-\zeta^2/4)\, d\zeta + c_2$$

$$-\varepsilon c_1^2 \int_0^{\xi} \left[\exp(-\zeta^2/4) \int_0^{\zeta} \exp(-\eta^2/4) d\eta\right] d\zeta \,, \tag{10.68}$$

which is the same approximate solution obtained by Method II and by approximating the exact solution.

***Parameter c***

*i*) *Method I*

For parameter *c*, Method I yields

$$\frac{dx}{2t} = \frac{dt}{0} = \frac{du}{-x\left(u - \varepsilon\frac{u^2}{2}\right)} \,, \tag{10.69}$$

which produces the similarity variables

$$\xi = t \,, \qquad u = \frac{f(\xi)}{\exp\left(-\frac{x^2}{4\xi}\right) + \frac{\varepsilon}{2} f(\xi)} \,. \tag{10.70}$$

It is obvious that the approximate solution contains higher order approximations also and is not a proper first order approximation. Instead of substituting the above solution, for simplicity, the solution is first approximated keeping only first order terms and substituted into the original equation yielding

$$u = \frac{c_1}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right) - \frac{\varepsilon}{2}\frac{c_1^2}{t} \exp\left(-\frac{x^2}{2t}\right) . \tag{10.71}$$

*ii*) *Method II*

To calculate the approximate solutions for Method II, use the symmetries (10.30)-(10.33). Parameter $c$ yields

$$\frac{dx}{2t} = \frac{dt}{0} = \frac{du_0}{-xu_0} = \frac{du_1}{-xu_1 + \frac{x}{2}u_0^2}\,, \tag{10.72}$$

with similarity variable and final solutions

$$\xi = t\,, \quad u_0 = \frac{c_1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right), \quad u_1 = \frac{c_2}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right) - \frac{c_1^2}{2t}\exp\left(-\frac{x^2}{2t}\right). \tag{10.73}$$

The approximate solution is then

$$u = \frac{c_1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right) + \varepsilon\left(\frac{c_2}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right) - \frac{c_1^2}{2t}\exp\left(-\frac{x^2}{2t}\right)\right), \tag{10.74}$$

which is an admissible first order approximate solution of the original equation.

*iii*) *Method III*

To calculate the approximate solutions for Method III, use the symmetries (10.46)-(10.49). Parameter $c$ with $h = 0$ yields the first order solution

$$u_0 = \frac{c_1}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right). \tag{10.75}$$

Now, $h(x,t)$ for the next order is

$$h = \left(\frac{\partial u_0}{\partial x}\right)^2 = \frac{c_1^2 x^2}{4t^3}\exp(-x^2/2t)\,. \tag{10.76}$$

To satisfy (10.49) with the above $h$ function and $c = 1$ while all other parameters are taken zero, $\alpha(x,t)$ should be selected as

$$\alpha = \frac{xc_1^2}{2t}\exp(-x^2/2t)\,. \tag{10.77}$$

The determining equations are

$$\frac{dx}{2t} = \frac{dt}{0} = \frac{du_1}{-x\left(u_1 - \frac{c_1^2}{2t}\exp(-x^2/2t)\right)}\,, \tag{10.78}$$

which produces the solution

$$u_1 = \frac{c_2}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right) - \frac{c_1^2}{2t}\exp\left(-\frac{x^2}{2t}\right) \tag{10.79}$$

Substituting $u_0$ and $u_1$ into the perturbation expansion, the same approximate solution obtained by Method II, i.e. Equation (10.74) is retrieved. In summary, while Method I contains higher order terms, Method II and III produced identical admissible first order solutions.

Since the exact symmetries are rich for the potential Burgers equation, Method I also produced approximate solutions. For restricted symmetries of the creeping flow of a non-Newtonian fluid, Method I failed to produce some approximate solutions that can be obtainable by Methods II and III (Pakdemirli et al., 2004). For more discussions and results on the comparisons of the three methods, see Pakdemirli et al. (2004). A more theoretical basis for the comparisons of Method I and Method II were presented by Wiltshire (2006). In summary, among the three symmetry methods, Method III is mostly recommended. Some additional approximate symmetry methods which are proposed more recently will be discussed in the following chapter.

## 10.3. EXERCISES

**E.10.1.** Perform the details of the calculations of the exact and approximate symmetries given in Section 10.1 for the potential Burgers equation.

**E.10.2.** Calculate the exact and approximate symmetries by Methods I, II and III of the nonlinear ODE

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0$$

and compare your results obtained by different methods.

**E.10.3.** Calculate the exact and approximate symmetries by Methods I, II and III of the nonlinear ODE

$$\frac{d^2u}{dt^2} + u + \varepsilon u^2 = 0$$

and compare your results obtained by different methods.

**E.10.4.** Using the symmetries calculated in E.10.2, find some exact and approximate solutions of the nonlinear ODE

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0$$

and compare your results obtained by different methods.

**E.10.5.** Using the symmetries calculated in E.10.3, find some exact and approximate solutions of the nonlinear ODE

$$\frac{d^2u}{dt^2} + u + \varepsilon u^2 = 0$$

and compare your results obtained by different methods.