CHAPTER 9

BOUNDARY VALUE PROBLEMS

In the previous chapters, it is observed that a given differential equation may admit a number of symmetries. Different solutions corresponding to different symmetries are always possible. The practical problem is to choose the right solution for the physical problem at hand. But physical problems in the form of differential equations appear with constraints which are named as initial or boundary conditions. Many of the mathematical solutions produced with different symmetry generators would then be useless as it comes to satisfy the initial/boundary conditions.

For nonlinear partial differential equations, in addition for the equation to be invariant under the given symmetry, the boundaries and the boundary conditions should also be invariant under the same transformation (Bluman and Kumei, 1989). This restricts the symmetries and usually a few of the symmetries may survive after the application of the symmetry generators to the boundaries and boundary conditions. What happens when a symmetry is admitted by the equation but not by the accompanying conditions? The answer is simple. For nonlinear PDEs, transformation of the equation in the form of similarity variables is possible but since the transformation of the conditions is not achieved, the reduced boundary value problem (BVP) cannot be expressed in terms of the new variables. The ideas will be exploited by treating sample problems.

BVP with infinite or semi-infinite domains are better in applying symmetry methods since such conditions impose no further restrictions on the symmetries. In the case of nonlinear PDEs, if the whole domain is finite, it may happen that all symmetries of the equations may become useless when the finite boundaries and conditions are imposed on the equations.

Another important problem is to find the appropriate boundary conditions for which the BVP transforms successfully to a system with reduced number of independent variables. To achieve this task, arbitrary functions may be introduced in the conditions and special forms for which the symmetry of the equation is admitted can then be determined. This task is similar to what is outlined in the previous chapter on group classifications. In the previous chapter, the group classification is maintained for parameters/functions appearing in the original equation itself whereas in this chapter, such arbitrary parameters/functions may be incorporated into the conditions.

The cases of ODEs or linear partial differential equations are somewhat different. For ODEs the main role of the symmetries is to introduce a reduction in the order of equation, hence, the transformation of conditions may not be necessary. For linear PDEs, there exist mechanisms to waive or loosen the restrictions on the symmetries due to the boundary conditions by applying superposition principle (Bluman and Kumei, 1989).

9.1. NONLINEAR PDES

As mentioned, for nonlinear PDEs, the boundaries and boundary conditions should also be invariant under the given symmetries. This puts severe restrictions on the symmetries and simplifies the Lie algebra much. Sometimes, one may have no symmetries at all after the invariance of boundaries and boundary conditions. A boundary condition at infinity however does not put additional restriction and can be ignored. This is the reason for similarity solutions being usually obtained for semi-infinite and infinite domains. For finite domains and for nonlinear PDEs, all symmetries may be lost. The boundary value problem for boundary layer equations of a Newtonian fluid is considered to exploit the ideas.

Problem 9.1. Consider the boundary value problem of the steady-state boundary layer flow of a Newtonian fluid (Pakdemirli and Yürüsoy, 1998)

$$u_x + v_y = 0 \,, \tag{9.1}$$

$$uu_x + vu_y = U(x)U'(x) + u_{yy} \,, \tag{9.2}$$

$$u(x,0) = 0 \,,\; v(x,0) = 0 \,,\;\; u(x,\infty) = U(x) \,. \tag{9.3}$$

Transform the system into an ODE system via similarity transformations.

Solution

The symmetries of the above equation were already calculated in Problem 6.6. The infinitesimals were

$$\xi_1 = ax + b\ , \tag{9.4}$$

$$\xi_2 = (a + c)y + d(x)\ , \tag{9.5}$$

$$\eta^1 = -(a + 2c)u\ , \tag{9.6}$$

$$\eta^2 = -(a + c)v + d'(x)u\ , \tag{9.7}$$

with the function $U(x)$ satisfying

$$(ax + b)\frac{d}{dx}(UU') + (3a + 4c)UU' = 0\ . \tag{9.8}$$

Applying the most general form of the generator $X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v}$ to the boundary $y = 0$ first,

$$Xy|_{y=0} = 0 \quad \rightarrow \quad \xi_2|_{y=0} = 0 \quad \rightarrow \quad d(x) = 0\ , \tag{9.9}$$

so that the infinite parameter Lie group of transformation is not suitable for the BVP. For the boundary and boundary condition at infinity, there is no need to require invariance. Next, apply the generator to the conditions

$$Xu|_{u=0} = 0 \quad \rightarrow \quad \eta^1|_{u=0} = 0\ , \tag{9.10}$$

$$Xv|_{v=0} = 0 \quad \rightarrow \quad \eta^2|_{v=0} = 0\ , \tag{9.11}$$

which are satisfied without further restrictions. Hence, the useful symmetries for the BVP are

$$\xi_1 = ax + b\ , \tag{9.12}$$

$$\xi_2 = (a + c)y, \tag{9.13}$$

$$\eta^1 = -(a + 2c)u\ , \tag{9.14}$$

$$\eta^2 = -(a + c)v, \tag{9.15}$$

$$(ax + b)\frac{d}{dx}(UU') + (3a + 4c)UU' = 0\ . \tag{9.16}$$

Ignoring the translational symmetry, i.e., $b = 0$, one may select

$$c = -\frac{m+1}{2}a\ , \tag{9.17}$$

without loss of generality so that the outer velocity can be expressed in a simpler form

$$U(x) = kx^m \ , \tag{9.18}$$

which satisfies (9.16). The infinitesimals are

$$\xi_1 = ax \ , \tag{9.19}$$

$$\xi_2 = \frac{1-m}{2} ay, \tag{9.20}$$

$$\eta^1 = mau \ , \tag{9.21}$$

$$\eta^2 = \frac{m-1}{2} av. \tag{9.22}$$

The characteristic equations together with (9.18)

$$\frac{dx}{x} = \frac{dy}{(\frac{1-m}{2})y} = \frac{du}{mu} = \frac{dv}{(\frac{m-1}{2})v} \ , \tag{9.23}$$

lead to the similarity variable and functions

$$\xi = yx^{(m-1)/2} \ , \quad u = x^m f(\xi) \ , \quad v = x^{(m-1)/2} g(\xi) \ , \quad U = kx^m \ . \tag{9.24}$$

The system successfully reduces to an ODE system

$$mf + \frac{m-1}{2}\xi f' + g' = 0 \ , \tag{9.25}$$

$$mf^2 + \frac{m-1}{2}\xi f f' + g f' = f'' + mk^2 \ , \tag{9.26}$$

$$f(0) = 0 \ , \quad g(0) = 0 \ , \ f(\infty) = k \ , \tag{9.27}$$

which can be integrated numerically. In fact, the ad hoc transformation of Prandtl (1905), founder of the boundary layer theory, was basically a kind of scaling transformation. Results presented here are in full agreement with the results given in Problem 2.8 for the special scaling transformation.

In fluid flow problems, a common symmetry is the scaling symmetry and if the equation of motion is too involved such as in the case of non-Newtonian fluids, a scaling symmetry test may retrieve useful solutions without proceeding further into the extensive algebra of general Lie group approach.

9.2. FINITE VERSUS INFINITE DOMAINS

The restrictions from finite domains on the symmetries are illustrated in the next problem.

Problem 9.2. Consider the dimensionless heat conduction (diffusion) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \tag{9.28}$$

with two different sets of conditions

$$u(0,t) = 1\,,\;\; u(\infty,t) = 0\,,\; u(x,0) = 0 \quad \text{(Cond. I)} \tag{9.29}$$

$$u(0,t) = 1\,,\;\; u(1,t) = 0\,,\; u(x,0) = 0 \quad \text{(Cond. II)} \tag{9.30}$$

Show that the scaling symmetry $X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ admitted by the equation is appropriate for the first set of conditions but not for the second set of conditions.

Solution

The above scaling symmetry and the similarity transformations were already obtained in Problem 2.4. The associated characteristic equation is

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{0}\quad , \tag{9.31}$$

with the similarity variables being

$$\xi = \frac{x}{\sqrt{t}}\,, \qquad u = F(\xi)\ . \tag{9.32}$$

Substituting into the original equation

$$F'' + \frac{\xi}{2}F' = 0\,, \tag{9.33}$$

the solution is

$$u(x,t) = c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\xi^2/4} d\xi + c_2\ . \tag{9.34}$$

i) *Conditions* I

Apply the generator to the boundaries

$$Xx|_{x=0} = 0 \quad \rightarrow \quad x|_{x=0} = 0 \quad \rightarrow \quad 0 = 0\,, \tag{9.35}$$

$$Xt|_{t=0} = 0 \quad \rightarrow \quad 2t|_{t=0} = 0 \quad \rightarrow \quad 0 = 0\,, \tag{9.36}$$

and then to the conditions

$$Xu|_{u=1} = X(1) \quad \rightarrow \quad 0 = 0\,, \tag{9.37}$$

$$Xu|_{u=0} = 0 \quad \rightarrow \quad 0 = 0\,, \tag{9.38}$$

Since all the conditions neglecting the one corresponding to the infinite domain has been satisfied, the solution will be appropriate for this set. Apply now the conditions

$$u(0,t) = c_1 \int_0^0 e^{-\xi^2/4} d\xi + c_2 = 1 \quad \rightarrow \quad c_2 = 1\,, \tag{9.39}$$

$$u(\infty,t) = c_1 \int_0^\infty e^{-\xi^2/4} d\xi + 1 = 0 \quad \rightarrow \quad c_1 = -\frac{1}{\sqrt{4\pi}}\,, \tag{9.40}$$

since $\int_0^\infty e^{-\xi^2/4} d\xi = \sqrt{4\pi}$. The last condition is satisfied identically

$$u(x,0) = -\frac{1}{\sqrt{4\pi}} \int_0^\infty e^{-\frac{\xi^2}{4}} d\xi + 1 = 0\,. \tag{9.41}$$

Hence, this set of conditions for semi-infinite domain is suitable for the given symmetry with the final solution being

$$u(x,t) = 1 - \frac{1}{\sqrt{4\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\xi^2/4} d\xi\,. \tag{9.42}$$

i) *Conditions* II

The finite domain set of conditions is not appropriate for the solution. To see this, check the invariance of the boundary $x = 1$,

$$Xx|_{x=1} = 0 \quad \rightarrow \quad x|_{x=1} = 0 \quad \rightarrow \quad 1 \neq 0\,, \tag{9.43}$$

The condition $u(x,1)$ cannot be satisfied by the solution

$$u(1,t) = c_1 \int_0^{\frac{1}{\sqrt{t}}} e^{-\xi^2/4} d\xi + c_2 \neq 0\,, \tag{9.44}$$

due to the above inappropriate transformation which is variable.

9.3. GROUP CLASSIFICATION FOR CONDITIONS

Since all symmetry solutions of the equations cannot be satisfied for a given set of initial/boundary conditions, one approach may be to determine for which set of conditions, the solutions can be satisfied by the conditions. To achieve this goal, arbitrary functions may be incorporated into the conditions and the special form of such arbitrary functions for which the similarity transformation may be appropriate is searched. The ideas are exploited in the following sample problem.

Problem 9.3. Consider the first order partial differential equation

$$u_t = u u_x \,, \tag{9.45}$$

for which the infinitesimals and generators $X = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}$ were already calculated in Problem 6.3.

$$\xi_1 = (at + b)x + ct^2 + dt + e \,, \tag{9.46}$$

$$\xi_2 = (cx + f)t + ax^2 + gx + h \,, \tag{9.47}$$

$$\eta = -cx - f + (ax - ct + g - d)u + (at + b)u^2 \,. \tag{9.48}$$

For the set of conditions

$$u(x,0) = A(x) \,, \;\; u(0,t) = 0 \,, \tag{9.49}$$

find the restricted infinitesimals and the classifying relation for the arbitrary function $A(x)$. Using special cases of the symmetries, find the special form of $A(x)$ that is appropriate. Give some sample symmetries and their solutions for which (9.49) is inappropriate at all.

Solution

Apply the generator to the boundaries

$$Xx|_{x=0} = 0 \quad \rightarrow \;\; \xi_2|_{x=0} = 0 \;\; \rightarrow \quad ft + h = 0 \quad \rightarrow \quad f = h = 0 \,, \tag{9.50}$$

$$Xt|_{t=0} = 0 \quad \rightarrow \;\; \xi_1|_{t=0} = 0 \;\; \rightarrow \quad bx + e = 0 \quad \rightarrow \quad b = e = 0 \,, \tag{9.51}$$

and then to the second condition

$$Xu|_{x=0,u=0} = 0 \quad \rightarrow \;\; \eta|_{x=0,u=0} = 0 \;\; \rightarrow \;\; 0 = 0 \;\; \rightarrow \quad \text{satisfied} \,. \tag{9.52}$$

The invariance of the first condition requires

$$Xu|_{t=0,u=A} = XA \quad \rightarrow \quad \eta|_{t=0,u=A} = \xi_2|_{t=0}A' \,, \tag{9.53}$$

which leads to

$$-cx + (ax + g - d)A = (ax^2 + gx)A' \,. \tag{9.54}$$

Hence, the restricted symmetries under conditions (9.49) are

$$\xi_1 = atx + ct^2 + dt \,, \tag{9.55}$$

$$\xi_2 = cxt + ax^2 + gx \,, \tag{9.56}$$

$$\eta = -cx + (ax - ct + g - d)u + atu^2 \,. \tag{9.57}$$

where (9.54) is the classifying relation for the function $A(x)$. If one requires $A(x)$ to be arbitrary, then from (9.54) $a = c = d = g = 0$ with $\xi_1 = 0$, $\xi_2 = 0$ and $\eta = 0$ and one is left with no symmetries. However, for special forms of $A(x)$, if (9.54) is satisfied, then one can construct an admissible BVP in terms of the similarity variables.

i) *Parameter a*

If parameter a is considered with all other parameters being zero, then from (9.54)

$$xA = x^2A', \tag{9.58}$$

with a solution

$$A = kx. \tag{9.59}$$

The characteristic equations

$$\frac{dt}{xt} = \frac{dx}{x^2} = \frac{du}{xu + tu^2} \quad , \tag{9.60}$$

lead to

$$\xi = \frac{x}{t}, \qquad u = \frac{xf(\xi)}{1 - tf(\xi)} \,. \tag{9.61}$$

Substituting into the original PDE

$$f'(\xi) = 0 \quad \rightarrow \quad f = k \, . \tag{9.62}$$

The first condition $u(x,0) = A(x)$ transforms into $f(\infty) = k$ and the second condition is satisfied without further restriction. The final solution is

$$u = \frac{kx}{1-kt} \, . \tag{9.63}$$

ii) *Parameters g and d*

If the parameters g and d are both considered with all other parameters being zero, then from (9.54)

$$(g-d)A = gxA', \tag{9.64}$$

with a solution

$$A = kx^{1-\frac{d}{g}}. \tag{9.65}$$

The characteristic equations

$$\frac{dt}{td} = \frac{dx}{xg} = \frac{du}{u(g-d)} \quad , \tag{9.66}$$

lead to

$$\xi = \frac{x}{t^m}, \qquad u = x^{\frac{m-1}{m}} f(\xi) \, . \tag{9.67}$$

where $m = g/d$ and hence $A = kx^{\frac{m-1}{m}}$. Substituting into the original PDE and conditions

$$\xi^{\frac{m-1}{m}} ff' + m\xi f' + \frac{m-1}{m\xi^{1/m}} f^2 = 0 \, , \qquad f(\infty) = k \; , \tag{9.68}$$

which is a suitable transformation. For the similarity transformation corresponding to $m = 1$, see Problem 7.1.

iii) *Parameter f*

Since f is not a symmetry of the BVP, (See Eq. (9.49)), it is expected that the conditions will not transform for this special choice.

The characteristic equations

$$\frac{dt}{0}=\frac{dx}{ft}=\frac{du}{-f} \quad , \tag{9.69}$$

leads to

$$\mu = t\ , \qquad u=-\frac{x}{t}+g(\mu) \quad . \tag{9.70}$$

The equation transforms into

$$\mu g' + g = 0\ , \tag{9.71}$$

with a solution

$$g=\frac{k}{\mu},\ \ u=\frac{k-x}{t} \quad . \tag{9.72}$$

However, $u(x,0)=A(x)$, does not transform properly since $\frac{k-x}{0}=\infty \neq A(x)$.

iv) *Parameter b*

An interesting case is parameter b. Although it is not a symmetry of the BVP, the equation and the conditions transform for this case. Skipping the details, the similarity variables are

$$\mu = x\ , \qquad u=\frac{x}{xg(x)-t}\ . \tag{9.73}$$

Substituting into the original equation, $g(x)=k/x$ and the solution is

$$u=\frac{x}{k-t}\ . \tag{9.74}$$

For $u(x,0)=A(x)$, the condition is satisfied if $A(x)=\frac{x}{k}$ and the second condition $u(0,t)=0$ is already satisfied.

9.4. EXERCISES

E.9.1. For the mathematical model of nonlinear fin equation

$$\theta_{xx}-N^2\theta=\theta_t\ ,$$

the infinitesimals for the generator $X=\xi_1\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial\theta}$ were calculated

$$\xi_1 = axt+\frac{1}{2}bx+dt+e\ ,$$

$$\xi_2 = at^2 + bt + c \ ,$$

$$\eta = \left(-\tfrac{1}{4}ax^2 - \tfrac{1}{2}dx - \tfrac{1}{2}at - (at^2 + bt)N^2 + h\right)\theta + g(x,t) \ ,$$

with

$$g_t = g_{xx} - N^2 g \ .$$

For the set of conditions

$$\theta(x,0) = 0 \ , \ \ \theta(0,t) = P(t), \quad \theta(\infty,t) = 0 \ ,$$

a) Find the restricted infinitesimals,

b) Find the classifying relation for $P(t)$,

c) Find a similarity solution from the restricted infinitesimals for a special $P(t)$ satisfying the classifying relation,

d) If the last condition is changed to $\theta(1,t) = 0$, can one find a similarity solution?

E.9.2. For the mathematical model of boundary layer flow of the special non-Newtonian fluid over a stretching sheet

$$u_x + v_y = 0 \ ,$$

$$uu_x + vu_y = u_{yy} + 6ku_{yy}u_y^2,$$

the infinitesimals for the generator $X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v}$ were given (Yürüsoy and Pakdemirli, 1999a)

$$\xi_1 = 3ax + b \ ,$$

$$\xi_2 = ay + d(x) \ ,$$

$$\eta^1 = au \ ,$$

$$\eta^2 = -av + d'(x)u \ .$$

For the set of conditions

$$u(x,0) = A(x) \ , \ \ v(x,0) = 0, \quad u(x,\infty) = 0 \ ,$$

a) Find the restricted infinitesimals,

b) Find the classifying relation for $A(x)$,

c) Perform a similarity transformation to reduce the PDE into an ODE system,

d) If the last condition is changed to $u(x,1) = 0$, can one find a non-trivial similarity solution?