CHAPTER 8

GROUP CLASSIFICATION PROBLEMS

Generalization of differential equations to cover a wide range of problems by insertion of arbitrary functions or parameters other than the dependent variables of the system is a common task performed in mathematical modeling. The specific forms of these parameters/functions represent special solutions for the problem. Some special forms may inherit more exact analytical solutions than the others. To determine those special forms which lead to more analytical solutions then constitute a group classification problem in the context of symmetry methods.

The specific value of parameters or the specific forms of functions may have direct consequences on the number of symmetry base generators admitted by the equations. If the parameters/functions are required to be arbitrary, that is in their most general forms, then, minimal numbers of symmetries are retrieved. The base symmetry generators corresponding to these minimum numbers of symmetries are named as the principle Lie algebra. When the parameters/functions attain special forms, extra symmetry generators are added to the Lie algebra which is called as the extensions of the principal Lie algebra. Only the case of arbitrary parameters/functions appearing in the equations will be treated in this chapter. Such arbitrary parameters/functions appearing in the boundary conditions will be delayed until next chapter on boundary value problems.

8.1. ORDINARY DIFFERENTIAL EQUATIONS

A sample ODE from the area of dynamical motion will be treated in this section.

Problem 8.1. The differential equation describing the path of the minimum drag work of a flying object is (Pakdemirli and Aksoy, 2010; Pakdemirli, 2009)

$$y'' - \frac{f'(y)}{f(y)}(1 + y'^2) = 0 \;, \tag{8.1}$$

where $y = y(x)$ is the altitude and $f(y) = \rho(y)C_d(y)A(y)U^2(y)$. Some or all of the physical parameters, namely the density ρ, the drag coefficient C_d, the cross-sectional area A and the velocity of the object U are assumed to be functions of altitude. Perform a group classification analysis with respect to the arbitrary function $f(y)$ and determine the principle Lie algebra and its extensions.

Solution

Defining the higher order variables

$$y_1 = y' \ , \ \ y_2 = y'' \ , \tag{8.2}$$

and rewriting (8.1)

$$F(y, y_1, y_2) = y_2 - \frac{f'(y)}{f(y)}(1 + y_1^2) = 0 \ . \tag{8.3}$$

The extended generator to second order is

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \eta^{(2)}(x,y,y_1,y_2)\frac{\partial}{\partial y_2} \ , \tag{8.4}$$

where the extended infinitesimals were given in Equations (4.23) and (4.24)

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y_1 - \xi_y y_1^2 \ , \tag{8.5}$$

$$\begin{aligned}\eta^{(2)} = {} & \eta_{xx} + (2\eta_{xy} - \xi_{xx})y_1 + (\eta_{yy} - 2\xi_{xy})y_1^2 - \xi_{yy}y_1^3 \\ & + (\eta_y - 2\xi_x)y_2 - 3\xi_y y_1 y_2 \ .\end{aligned} \tag{8.6}$$

Applying the invariance condition $XF \equiv 0$,

$$\eta^{(2)} - \frac{f'(y)}{f(y)}2y_1\eta^{(1)} - (1 + y_1^2)\eta\left(\frac{f'(y)}{f(y)}\right)' = 0 \ . \tag{8.7}$$

Substituting (8.5) and (8.6) into (8.7) and using $y_2 = \frac{f'(y)}{f(y)}(1 + y_1^2)$ wherever it appears, the system is a block of equations which can be separated with respect to y_1

$$\eta_{xx} + (\eta_y - 2\xi_x)\frac{f'(y)}{f(y)} - \eta\left(\frac{f'(y)}{f(y)}\right)' = 0 \ , \tag{8.8}$$

$$2\eta_{xy} - \xi_{xx} - \left(3\xi_y + 2\eta_x\right)\frac{f'(y)}{f(y)} = 0 \ , \tag{8.9}$$

$$\eta_{yy} - 2\xi_{xy} - \eta_y \frac{f'(y)}{f(y)} - \eta\left(\frac{f'(y)}{f(y)}\right)' = 0 \ , \tag{8.10}$$

$$\xi_{yy} + \xi_y \frac{f'(y)}{f(y)} = 0 \ . \tag{8.11}$$

The over determined system is solved next. Starting from (8.11) and integrating

$$\xi = a(x)\int \frac{dy}{f(y)} + b(x) \ . \tag{8.12}$$

Equation (8.10) is integrated with respect to y

$$\eta_y - 2\xi_x - \eta \frac{f'(y)}{f(y)} = c'(x) \ . \tag{8.13}$$

Solving the above equation for η

$$\eta = \left(c'(x) + 2b'(x)\right)f(y)\int \frac{dy}{f(y)} + 2a'(x)f(y)\int\left(\frac{1}{f(y)}\int \frac{dy}{f(y)}\right)dy + d(x)f(y). \tag{8.14}$$

Substitute the equivalents of ξ and η into (8.9)

$$2c''(x) + 3b''(x) + 3a''(x)\int \frac{dy}{f(y)} - 3a(x)\frac{f'(y)}{f^2(y)} = 0 \ . \tag{8.15}$$

Differentiating the above equation with respect to y yields the classifying relation

$$a''(x)\frac{1}{f(y)} - a(x)\left(\frac{f'(y)}{f^2(y)}\right)' = 0 \ . \tag{8.16}$$

Several cases should be distinguished before proceeding further.

***i*) Arbitrary $f(y)$**

For arbitrary function, from (8.16), $a(x) = 0$ and the infinitesimals satisfying (8.8)-(8.11) turns out to be

$$\xi = a \ , \qquad \eta = 0 \ . \tag{8.17}$$

The one-parameter Lie group of transformation is the minimal number which constitutes the principle Lie algebra.

***ii*) $\boldsymbol{f(y) = 1/(k_1 y + k_2)}$**

This choice corresponds to $\left(\frac{f'(y)}{f^2(y)}\right)' = 0$. From (8.16), $a = a_1 x + a_2$. After some algebra (8.8)-(8-11) yields

$$\xi = (ax + b)\left(\frac{1}{2}k_1 y^2 + k_2 y\right) + \frac{1}{2}k_1 a x^3 + \frac{1}{2}cx^2 + dx + e\,, \tag{8.18}$$

$$\eta = \left(\frac{1}{2}(c - 3k_1 b)x + g + 2d\right)\frac{\frac{1}{2}k_1 y^2 + k_2 y}{k_1 y + k_2} + a\frac{\frac{1}{4}k_1^2 y^4 + k_1 k_2 y^3 + k_2^2 y^2}{k_1 y + k_2}$$
$$+ \frac{-\frac{1}{4}k_1^2 a x^4 - \frac{1}{4}k_1(c + k_1 b)x^3 + \frac{1}{2}k_1 g x^2 + hx + \ell}{k_1 y + k_2}\,. \tag{8.19}$$

For this specific choice, the Lie algebra extended from 1-parameter to 8-parameter Lie group of transformations which is the maximum number that can be attained for second order ODEs.

***iii*) $\boldsymbol{f(y) = k}$**

For this choice, for which the function is a constant, Equations (8.8)-(8-11) yields

$$\xi = axy + by + cx^2 + dx + e\,, \tag{8.20}$$

$$\eta = cxy + gy + ay^2 + hx + \ell\,, \tag{8.21}$$

which is again an 8-parameter Lie group of transformations.

***iv*) $\boldsymbol{f(y) = k\exp(\alpha y)}$**

For this choice, the infinitesimals are

$$\xi = (a\cos\alpha x + b\sin\alpha x)e^{-\alpha y} + c\cos 2\alpha x + d\sin 2\alpha x + e\,, \tag{8.22}$$

$$\eta = (a\sin\alpha x - b\cos\alpha x)e^{-\alpha y} + c\sin 2\alpha x - d\cos 2\alpha x + g$$
$$+(h\cos\alpha x + \ell\sin\alpha x)e^{\alpha y}\,, \tag{8.23}$$

which contains 8 parameters. Results are summarized in Table 8.1 for convenience.

Table 8.1. Group Classification Results for Problem 8.1

Function	Symmetries
Arbitrary $f(y)$	$\xi = a$ $\eta = 0$
$f(y) = 1/(k_1 y + k_2)$	$\xi = (ax + b)\left(\frac{1}{2}k_1 y^2 + k_2 y\right) + \frac{1}{2}k_1 a x^3 + \frac{1}{2}cx^2 + dx + e$ $\eta = \left(\frac{1}{2}(c - 3k_1 b)x + g + 2d\right)\frac{\frac{1}{2}k_1 y^2 + k_2 y}{k_1 y + k_2} + a\frac{\frac{1}{4}k_1^2 y^4 + k_1 k_2 y^3 + k_2^2 y^2}{k_1 y + k_2}$ $+ \frac{-\frac{1}{4}k_1^2 a x^4 - \frac{1}{4}k_1(c + k_1 b)x^3 + \frac{1}{2}k_1 g x^2 + hx + \ell}{k_1 y + k_2}$
$f(y) = k$	$\xi = axy + by + cx^2 + dx + e$ $\eta = cxy + gy + ay^2 + hx + \ell$
$f(y) = k\exp(\alpha y)$	$\xi = (a\cos\alpha x + b\sin\alpha x)e^{-\alpha y} + c\cos 2\alpha x + d\sin 2\alpha x + e$ $\eta = (a\sin\alpha x - b\cos\alpha x)e^{-\alpha y} + c\sin 2\alpha x - d\cos 2\alpha x + g$ $+(h\cos\alpha x + \ell\sin\alpha x)e^{\alpha y}$

For group invariant solutions and reductions of order using the above symmetries, see Pakdemirli and Aksoy (2010).

8.2. PARTIAL DIFFERENTIAL EQUATIONS

Two sample PDE systems from boundary layer theory will be discussed in this section.

Problem 8.2. Consider the non-dimensional steady state boundary layer flow of a Newtonian fluid modeled by the coupled equations (Pakdemirli and Yürüsoy, 1998)

$$u_x + v_y = 0 \;, \tag{8.24}$$

$$uu_x + vu_y = U(x)U'(x) + u_{yy} \;, \tag{8.25}$$

Perform the group classification with respect to the outer velocity $U(x)$.

Solution

The symmetries of the above equation were already calculated in Problem 6.6 and therefore the calculations are not repeated. The infinitesimals were

$$\xi_1 = ax + b \;, \tag{8.26}$$

$$\xi_2 = (a + c)y + d(x) \;, \tag{8.27}$$

$$\eta^1 = -(a + 2c)u \;, \tag{8.28}$$

$$\eta^2 = -(a+c)v + d'(x)u\,, \tag{8.29}$$

with the function $U(x)$ satisfying

$$(ax+b)\frac{d}{dx}(UU') + (3a+4c)UU' = 0\,. \tag{8.30}$$

The generators were

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \tag{8.31}$$

In fact, Equation (8.30) is the classifying relationship. Several distinct cases have to be treated.

For an arbitrary outer velocity, it is impossible to satisfy (8.30) unless $a = b = c = 0$. Hence only the infinite parameter generator survives

$$X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\,, \tag{8.32}$$

which is the principle Lie algebra for the equation. To find the other specific cases, (8.30) can be integrated directly

$$\frac{d(UU')}{UU'} = -\frac{3a+4c}{ax+b}dx \qquad , \tag{8.33}$$

or

$$UU' = \bar{k}_1(ax+b)^{-3-4\frac{c}{a}}\,, \tag{8.34}$$

which can be further integrated by separation of variables. The solution is

$$U = k_1(ax+b)^m\ ,\ m = -1 - 2\frac{c}{a}\,, \tag{8.35}$$

where the first integration constant is redefined and the second integration constant has been selected as zero without loss of generality. Since all parameters survive in this case, the generators are the full group given by (8.31).

A special case occurs when $c = -\frac{a}{2}$. From (8.33)

$$\frac{d(UU')}{UU'} = -\frac{a}{ax+b}dx \ . \tag{8.36}$$

Integrating two times and taking the second integration constant as zero, the final result is

$$U = k\sqrt{\ln(ax+b)}\ ,\ c = -\frac{a}{2} \ . \tag{8.37}$$

The infinitesimals and the generators for this case are

$$\xi_1 = ax+b,\ \xi_2 = \frac{a}{2}y + d(x),\ \eta^1 = 0,\ \eta^2 = -\frac{a}{2}v + d'(x)u\ , \tag{8.38}$$

$$X_1 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} - \frac{v}{2}\frac{\partial}{\partial v},\ \ X_2 = \frac{\partial}{\partial x},\ X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v} \ . \tag{8.39}$$

which has one less generator.

If $a = 0$, from (8.33)

$$\frac{d(UU')}{UU'} = -\frac{4c}{b}dx \ , \tag{8.40}$$

and integrating twice with taking the second integration constant as zero

$$U = ke^{mx}\ ,\ m = -2\frac{c}{b} \ . \tag{8.41}$$

The infinitesimals and the generators for this case are

$$\xi_1 = b,\ \xi_2 = cy + d(x),\ \eta^1 = -2cu,\ \eta^2 = -cv + d'(x)u\ , \tag{8.42}$$

$$X_2 = \frac{\partial}{\partial x},\ X_3 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},\ X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \tag{8.43}$$

If $c = 0$, from (8.35), $m = -1$, and

$$U = \frac{k}{x+n}\ ,\ n = \frac{b}{a} \ . \tag{8.44}$$

The infinitesimals and the generators for this case are

$$\xi_1 = ax+b,\ \xi_2 = ay + d(x),\ \eta^1 = -au,\ \eta^2 = -av + d'(x)u\ , \tag{8.45}$$

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},\ X_2 = \frac{\partial}{\partial x},\ X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \tag{8.46}$$

If $b = 0$, from (8.35),

$$U = kx^m\ ,\ m = -1 - 2\frac{c}{a}\ , \tag{8.47}$$

and the infinitesimals and generators are

$$\xi_1 = ax,\ \xi_2 = (a+c)y + d(x),\ \eta^1 = -(a+2c)u,$$
$$\eta^2 = -(a+c)v + d'(x)u\ , \tag{8.48}$$

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},\quad X_3 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$
$$X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \tag{8.49}$$

If

$$U = k\ , \tag{8.50}$$

equation (8.30) is satisfied and all symmetries survive.

Finally, another special case is $c = -\frac{3}{4}a$ for which (8.30) yields

$$U = k\sqrt{x}\ , \tag{8.51}$$

and the infinitesimals and generators are

$$\xi_1 = ax + b,\ \xi_2 = \frac{a}{4}y + d(x),\ \eta^1 = \frac{a}{2}u,\ \eta^2 = -\frac{a}{4}v + d'(x)u\ , \tag{8.52}$$

$$X_1 = x\frac{\partial}{\partial x} + \frac{y}{4}\frac{\partial}{\partial y} - \frac{u}{2}\frac{\partial}{\partial u} - \frac{v}{4}\frac{\partial}{\partial v},\quad X_2 = \frac{\partial}{\partial x},\quad X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \tag{8.53}$$

Results are summarized in Table 8.2 for convenience.

Table 8.2. Group Classification Results for Problem 8.2

Function	Parameters	Symmetries
Arbitrary $U(x)$	$a=b=c=0$	$\xi_1=0$ $\xi_2=d(x)$ $\eta^1=0$ $\eta^2=d'(x)u$
$U=k_1(ax+b)^m$	$m=-1-2\dfrac{c}{a}$	$\xi_1=ax+b$ $\xi_2=(a+c)y+d(x)$ $\eta^1=-(a+2c)u$ $\eta^2=-(a+c)v+d'(x)u$
$U=k\sqrt{\ln(ax+b)}$	$c=-\dfrac{a}{2}$	$\xi_1=ax+b$ $\xi_2=\dfrac{a}{2}y+d(x)$ $\eta^1=0$ $\eta^2=-\dfrac{a}{2}v+d'(x)u$
$U=ke^{mx}$	$a=0$ $m=-2\dfrac{c}{b}$	$\xi_1=b$ $\xi_2=cy+d(x)$ $\eta^1=-2cu$ $\eta^2=-cv+d'(x)u$
$U=\frac{k}{x+n}$,	$c=0$ $n=\dfrac{b}{a}$	$\xi_1=ax+b$ $\xi_2=ay+d(x)$ $\eta^1=-au$ $\eta^2=-av+d'(x)u$
$U=kx^m$	$b=0$ $m=-1-2\dfrac{c}{a}$	$\xi_1=ax$ $\xi_2=(a+c)y+d(x)$ $\eta^1=-(a+2c)u$ $\eta^2=-(a+c)v+d'(x)u$
$U=k$		$\xi_1=ax+b$ $\xi_2=(a+c)y+d(x)$ $\eta^1=-(a+2c)u$ $\eta^2=-(a+c)v+d'(x)u$
$U=k\sqrt{x}$	$c=-\dfrac{3}{4}a$	$\xi_1=ax+b$ $\xi_2=\dfrac{a}{4}y+d(x)$ $\eta^1=\dfrac{a}{2}u$ $\eta^2=-\dfrac{a}{4}v+d'(x)u$

Problem 8.3. Consider the steady state two dimensional boundary layer flow of a non-Newtonian fluid (Yürüsoy and Pakdemirli, 1999b)

$$u_x+v_y=0\ , \tag{8.54}$$

$$uu_x+vu_y=T(u_y)u_{yy}+f(x)\ , \tag{8.55}$$

where u and v are the x and y components of velocity inside the boundary layer. The functions are defined as

$$T(u_y) = \tau'(u_y)\ , \tag{8.56}$$

$$f(x) = U(x)U'(x), \tag{8.57}$$

where τ is the shear stress and T is the derivative of the shear stress with respect to its argument, namely the velocity gradient. $U(x)$ is the outer velocity. The importance of the model is that it covers a number of fluids such as Newtonian, power-law, Powel-Eyring, Prandl and Williamson. Perform the group classification with respect to the arbitrary $T(u_y)$ function.

Solution

Defining $x_1 = x$, $x_2 = y$, $u^1 = u$, $u^2 = v$, $u_1^1 = u_x$, $u_2^1 = u_y$, $u_1^2 = v_x$, $u_2^2 = v_y$ and $u_{22}^1 = u_{yy}$, the equations are

$$u_1^1 + u_2^2 = 0\ , \tag{8.58}$$

$$u^1u_1^1 + u^2u_2^1 = T(u_2^1)u_{22}^1 + f(x_1)\ . \tag{8.59}$$

The infinitesimal generator extended to second order is

$$X^{(2)} = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + \eta^1\frac{\partial}{\partial u^1} + \eta^2\frac{\partial}{\partial u^2} + \eta_1^{(1)1}\frac{\partial}{\partial u_1^1} + \eta_2^{(1)1}\frac{\partial}{\partial u_2^1}$$
$$+\eta_2^{(1)2}\frac{\partial}{\partial u_2^2} + \eta_{22}^{(2)1}\frac{\partial}{\partial u_{22}^1}\quad . \tag{8.60}$$

The extended infinitesimals are given in (6.141)-(6.143) and (6.146) for two independent and two dependent variables.

The invariance conditions are

$$\eta_1^{(1)1} + \eta_2^{(1)2} = 0\ , \tag{8.61}$$

$$u_1^1\eta^1 + u_2^1\eta^2 + u^1\eta_1^{(1)1} + u^2\eta_2^{(1)1} = \eta_2^{(1)1}T'u_{22}^1 + T\eta_{22}^{(2)1} + \xi_1 f'\ . \tag{8.62}$$

Using $u_2^2 = -u_1^1$ and $u_{22}^1 = \frac{1}{T}(u^1u_1^1 + u^2u_2^1 - f)$ wherever they appear, the invariance conditions form two blocks of equations which can be separated with respect to higher order variables. With some simplifications, the determining equations are

$$\xi_1 = \xi_1(x_1),\ \ \xi_2 = \xi_2(x_1, x_2, u^1)\ ,\ \ \eta^1 = \eta^1(x_1, x_2, u^1)\ , \tag{8.63}$$

$$\frac{\partial \eta^1}{\partial x_1} + \frac{\partial \eta^2}{\partial x_2} = 0\ , \tag{8.64}$$

$$\frac{\partial \eta^1}{\partial u^1} - \xi_1'(x_1) - \frac{\partial \eta^2}{\partial u^2} + \frac{\partial \xi_2}{\partial x_2} = 0\ , \tag{8.65}$$

$$\frac{\partial \eta^2}{\partial u^1} - \frac{\partial \xi_2}{\partial x_1} = 0\ , \tag{8.66}$$

$$\begin{aligned}
&u^1 \frac{\partial \eta^1}{\partial x_1} + u^2 \frac{\partial \eta^1}{\partial x_2} + f \frac{\partial \eta^1}{\partial u^1} - 2f \frac{\partial \xi_2}{\partial x_2} - \xi_1 f' \\
&+ \left(-u^1 \frac{\partial \xi_2}{\partial x_1} + \eta^2 + u^2 \frac{\partial \xi_2}{\partial x_2} - 3f \frac{\partial \xi_2}{\partial u^1}\right) u_2^1 + 2u^2 \frac{\partial \xi_2}{\partial u^1} (u_2^1)^2 \\
&= \frac{T'}{T} \{-f \frac{\partial \eta^1}{\partial x_2} + \left(u^2 \frac{\partial \eta^1}{\partial x_2} - f \frac{\partial \eta^1}{\partial u^1} + f \frac{\partial \xi_2}{\partial x_2}\right) u_2^1 \\
&+ \left[u^2 \left(\frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi_2}{\partial x_2}\right) + f \frac{\partial \xi_2}{\partial u^1}\right] (u_2^1)^2 - u^2 \frac{\partial \xi_2}{\partial u^1} (u_2^1)^3 \Big\} \\
&+ T \left\{\frac{\partial^2 \eta^1}{\partial x_2^2} + \left[2 \frac{\partial^2 \eta^1}{\partial x_2 \partial u^1} - \frac{\partial^2 \xi_2}{\partial x_2^2}\right] u_2^1 + \left[\frac{\partial^2 \eta^1}{\partial (u^1)^2} - 2 \frac{\partial^2 \xi_2}{\partial x_2 \partial u^1}\right] (u_2^1)^2 \right. \\
&\left. - \frac{\partial^2 \xi_2}{\partial (u^1)^2} (u_2^1)^3 \right\} ,
\end{aligned} \tag{8.67}$$

$$\begin{aligned}
&\eta^1 - u^1 \frac{\partial \xi_1}{\partial x_1} + 2u^1 \frac{\partial \xi_2}{\partial x_2} + 2u^1 \frac{\partial \xi_2}{\partial u^1} u_2^1 \\
&= \frac{T'}{T} \left\{u^1 \frac{\partial \eta^1}{\partial x_2} + u^1 \left(\frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi_2}{\partial x_2}\right) u_2^1 - u^1 \frac{\partial \xi_2}{\partial u^1} (u_2^1)^2 \right\} .
\end{aligned} \tag{8.68}$$

Separation with respect to u_2^1 did not occur since the function T depends on u_2^1. The simpler of the last two equations, namely (8.68) is selected as the classifying relation. Defining

$$\eta^1 - u^1 \frac{\partial \xi_1}{\partial x_1} + 2u^1 \frac{\partial \xi_2}{\partial x_2} = c_1 \lambda\ , \tag{8.69}$$

$$u^1 \frac{\partial \xi_2}{\partial u^1} = c_2 \lambda\ , \tag{8.70}$$

$$u^1 \frac{\partial \eta^1}{\partial x_2} = c_3 \lambda \ , \tag{8.71}$$

$$u^1 \left(\frac{\partial \eta^1}{\partial u^1} - \frac{\partial \xi_2}{\partial x_2} \right) = c_4 \lambda \ , \tag{8.72}$$

where $\lambda = \lambda(x_1, x_2, u^1)$ is an arbitrary function. From (8.68) then,

$$\frac{T'}{T} = \frac{c_1 + 2c_2 u_2^1}{c_3 + c_4 u_2^1 - c_2 \left(u_2^1\right)^2} \ . \tag{8.73}$$

The basic functions that admit the above equation are

$i)\ T = \nu$ (8.74)

$ii)\ T = \nu (u_2^1)^n$ (8.75)

$iii)\ T = \nu e^{\alpha u_2^1}$ (8.76)

$iv)\ T = \frac{\nu e^{(\sigma \arctan u_2^1)}}{1 + \left(u_2^1\right)^2}$ (8.77)

To find the principle Lie algebra, as an initial step, T is assumed to be arbitrary which enables separation of (8.67) and (8.68). Solving (8.64)-(8.68), returning back to the original variables, the infinitesimals are

$$\xi_1 = 3ax + b \ , \tag{8.78}$$

$$\xi_2 = ay + d(x) \ , \tag{8.79}$$

$$\eta^1 = au \ , \tag{8.80}$$

$$\eta^2 = -av + ud'(x) \ , \tag{8.81}$$

with the function $f(x)$ satisfying

$$(3ax + b)f' + af = 0 \quad . \tag{8.82}$$

The principle Lie algebra consists of two finite parameter Lie group of transformations and an infinite parameter Lie group of transformations if $f(x)$ satisfies (8.82) with nonzero parameters a and b. In fact, the group classification can be pursued to include $f(x)$ also which has been already outlined in the previous problem and skipped here for compactness. Hence, for an arbitrary

$f(x)$, parameters a and b also vanishes leaving with only the infinite symmetry generator. The generators corresponding to (8.78)-(8.81) are

$$X_1 = 3x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},\ X_2 = \frac{\partial}{\partial x},\ X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \quad (8.83)$$

The principle Lie Algebra does not extend for cases (*iii*) and (*iv*) due to the further restrictions from (8.67). Only for cases (*i*) and (*ii*), extension of the Lie algebra is achieved:

***i*) $\boldsymbol{T = \nu}$**

$$\xi_1 = ax + b\ , \quad (8.84)$$

$$\xi_2 = cy + d(x)\ , \quad (8.85)$$

$$\eta^1 = (a - 2c)u\ , \quad (8.86)$$

$$\eta^2 = -cv + ud'(x)\ , \quad (8.87)$$

with the function $f(x)$ satisfying

$$(ax + b)f' + (4c - a)f = 0\quad . \quad (8.88)$$

Since, this choice corresponds to a Newtonian fluid, results should agree with the ones given for Newtonian fluids. In fact, results are in full agreement with the ones given in (8.26)-(8.30) if the parameters are redefined as $\bar{a} = a,\ \bar{b} = b,\ \bar{d} = d,\ \bar{c} = a + c$ with overbars representing the parameters in (8.84)-(8.88). The group classification with respect to *f* using (8.88) is skipped and left to the reader. The generators for this case are

$$X_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\ X_2 = \frac{\partial}{\partial x}, X_3 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$

$$X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \quad (8.89)$$

The Lie algebra extended by one for Newtonian fluids.

***ii*) $\boldsymbol{T = \nu(u_y)^n\ (n \neq -1)}$**

$$\xi_1 = [(n-1)a + (n+2)b]x + c\ , \quad (8.90)$$

$$\xi_2 = by + d(x)\ , \quad (8.91)$$

$$\eta^1 = -au\ , \tag{8.92}$$

$$\eta^2 = -[na + (n+1)b]v + ud'(x)\ , \tag{8.93}$$

with the function $f(x)$ satisfying

$$\{(n-1)a + (n+2)b]x + c\}f' + \{(n+1)a + (n+2)b\}f = 0\ . \tag{8.94}$$

This choice corresponds to a power-law fluid. The group classification with respect to f using (8.94) is skipped and left to the reader. The generators for this case are

$$X_1 = (n-1)x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - nv\frac{\partial}{\partial v},\ , X_2 = (n+2)x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - (1+n)v\frac{\partial}{\partial v},$$

$$X_3 = \frac{\partial}{\partial x}\ , \qquad X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v}\ . \tag{8.95}$$

The Lie algebra extended again by one for power-law fluids.

Results are summarized in Table 8.3.

Table 8.3. Group Classification Results for Problem 8.3

Function	Classifying relation for $f(x)$	Symmetries
Arbitrary $T(u_y)$	$(3ax+b)f' + af = 0$	$\xi_1 = 3ax + b$ $\xi_2 = ay + d(x)$ $\eta^1 = au$ $\eta^2 = -av + ud'(x)$
$T = v$	$(ax+b)f' + (4c-a)f = 0$	$\xi_1 = ax + b$ $\xi_2 = cy + d(x)$ $\eta^1 = (a-2c)u$ $\eta^2 = -cv + ud'(x)$
$T = v(u_y)^n$	$\{(n-1)a + (n+2)b]x + c\}f'$ $+\{(n+1)a + (n+2)b\}f = 0$	$\xi_1 = [(n-1)a + (n+2)b]x + c$ $\xi_2 = by + d(x)$ $\eta^1 = -au$ $\eta^2 = -[na + (n+1)b]v + ud'(x)$

For similarity solutions of the model, see Pakdemirli (1994b) and Pakdemirli et al. (1996).

8.3. EXERCISES

E.8.1. Perform the group classification with respect to the constant acceleration ratio parameter ε for the constant acceleration curve equation (Pakdemirli, 2023b)

$$(1+y'^2)y''' - (3y' + 2\varepsilon)y''^2 = 0 \ .$$

E.8.2. Perform the group classification with respect to the heat conduction coefficient $k(u)$ for the nonlinear heat equation (Bluman and Kumei, 1989)

$$u_t = (k(u)u_x)_x \ .$$

E.8.3. Perform the group classification with respect to the heat conduction coefficient $k(\theta)$ and heat transfer coefficient $f(x)$ for the nonlinear fin equation (Pakdemirli and Şahin, 2004)

$$\theta_t = (k(\theta)\theta_x)_x - N^2 f(x)\theta \ .$$

E.8.4. Perform the group classification with respect to the filter coefficient $\lambda(\sigma)$ for the coupled nonlinear filtration equation (Pakdemirli, 2002)

$$c_x + \lambda(\sigma)c = 0 \ , \ \ \sigma_t + vc_x = 0 \ ,$$

where $c = c(x,t)$, $\sigma = \sigma(x,t)$ and v a constant.

E.8.5. Perform the group classification with respect to $f(x)$ for the steady state two dimensional boundary layer flow of a non-Newtonian fluid (Yürüsoy and Pakdemirli, 1999b)

$$u_x + v_y = 0 \ ,$$

$$uu_x + vu_y = T(u_y)u_{yy} + f(x) \ ,$$

for the special case of arbitrary $T(u_y)$.