CHAPTER 7

PARTIAL DIFFERENTIAL EQUATIONS: SYMMETRY SOLUTIONS

In the previous chapter, the infinitesimals and symmetry generators are calculated. In this chapter, the solutions using these symmetries will be discussed. For ODEs, it was shown in Chapter 5 that one-parameter Lie group symmetry reduces the order of the equation by one. In partial differential equations, the effect of symmetries is somewhat different. One-parameter symmetry reduces the number of independent variables by one. This is done by the so called similarity variables (group invariants) and the solutions are called similarity (group invariant) solutions. If the PDE has two independent variables, then employing one symmetry transforms the equation into an ODE which is easier to handle either analytically or numerically. It is always possible to achieve multiple reductions in the number of independent variables by employing more than one symmetry. The other option is to map a solution from a known solution using the symmetries. The requirement for this to happen is that the known solution should not be a group invariant solution, otherwise, the solution will map onto itself if the generator producing that solution is employed. More details will be given in the worked examples on the issue.

The algorithm to calculate similarity solutions is as follows. Given a scalar PDE

$$F\left(x_i, u, u_i, u_{ij}, u_{ijk}, \ldots\right) = 0 \ , \ i, j, k, .. = 1,2, \ldots n. \tag{7.1}$$

with a symmetry generator admitted by the equation

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} \quad . \tag{7.2}$$

The similarity variables are calculated from the characteristic equation

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \cdots = \frac{dx_n}{\xi_n} = \frac{du}{\eta} \, . \tag{7.3}$$

In fact, they are the invariants of the above system of equations. By direct substitution of the new variables into the original equation (7.1), the *n*-

independent variable PDE transforms into another PDE with n-1-independent variables. If $n=2$, then the transformed equation is an ODE.

7.1. FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

The first order PDE treated in the previous chapter is retreated again. The symmetries were already calculated. Group invariant solutions as well as mapping a solution into another solution will be considered.

Group Invariant Solutions

Problem 7.1. Consider the first order partial differential equation

$$u_t = uu_x \,, \tag{7.4}$$

for which the infinitesimals and generators were already calculated in Problem 6.3.

$$\xi_1 = (at+b)x + ct^2 + dt + e \,, \tag{7.5}$$

$$\xi_2 = (cx+f)t + ax^2 + gx + h \,, \tag{7.6}$$

$$\eta = -cx - f + (ax - ct + g - d)u + (at+b)u^2 \,, \tag{7.7}$$

$$X_1 = xt\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + (xu + tu^2)\frac{\partial}{\partial u}\,,\ X_2 = x\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u}\,,$$

$$X_3 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}\,, X_4 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\,, X_5 = \frac{\partial}{\partial t}\,,$$

$$X_6 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}\,, X_7 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\,,\ X_8 = \frac{\partial}{\partial x}\,. \tag{7.8}$$

Calculate

i) A group invariant solution corresponding to $g = d$ $(X_4 + X_7)$ with all other parameters being equal to zero.

ii) A group invariant solution corresponding to e and f $(eX_5 + fX_6)$ with all other parameters being equal to zero.

i) *Solution corresponding to parameters* $g = d$

For this choice $\xi_1 = t$, $\xi_2 = x$ and $\eta = 0$. The characteristic equation is,

$$\frac{dt}{t} = \frac{dx}{x} = \frac{du}{0} \,. \tag{7.9}$$

From the first two equations $\frac{dt}{t} = \frac{dx}{x}$,

$$\xi = \frac{x}{t} \,, \tag{7.10}$$

and from the last two equations $\frac{dx}{x} = \frac{du}{0}$,

$$u = f(\xi) \,. \tag{7.11}$$

Hence ξ and $f(\xi)$ are the similarity variables (group invariants). Calculate the derivatives

$$u_t = f'(\xi)\xi_t = -\frac{x}{t^2} f'(\xi), \qquad u_x = f'(\xi)\xi_x = \frac{1}{t} f'(\xi) \,, \tag{7.12}$$

and substitute into the original equation

$$-\frac{x}{t^2} f'(\xi) = f(\xi) \frac{1}{t} f'(\xi) \,. \tag{7.13}$$

Multiply the equation by *t* and use (7.10) to finally obtain the transformed ODE

$$f'(\xi)(f(\xi) + \xi) = 0 \,, \tag{7.14}$$

which yields $f = k_1$, a constant or $f = -\xi$ which are both group invariant solutions. Discarding the trivial constant solution, the nontrivial solution is $f = -\xi$, or

$$u = -\frac{x}{t} \,, \tag{7.15}$$

which definitely satisfies the original equation. Inspecting (7.14), one realizes that the transformed equation contains only similarity variables. If one does not use a symmetry generator of the equation, then the transformed equation does not appear purely in terms of the similarity variables.

ii) *Solution corresponding to parameters e and f*

For this choice $\xi_1 = e$, $\xi_2 = ft$ and $\eta = -f$. The characteristic equation is,

$$\frac{dt}{e} = \frac{dx}{ft} = \frac{du}{-f} \,. \tag{7.16}$$

From the first two equations, defining $\frac{f}{e} = 2k_1$,

$$\mu = x - k_1 t^2 \,, \tag{7.17}$$

and from the first and third ones,

$$u = -2k_1 t + g(\mu) \,. \tag{7.18}$$

Calculating the derivatives and inserting into the original PDE

$$g(\xi)g'(\xi) = -2k_1 \,, \tag{7.19}$$

which can be solved easily

$$g(\xi) = 2\sqrt{k_2 - k_1\mu} \,. \tag{7.20}$$

Returning back to the original variables, the solution is

$$u = -2k_1 t + 2\sqrt{k_2 - k_1(x - k_1 t^2)} \,. \tag{7.21}$$

which definitely satisfies the original equation. As outlined above, one can obtain many group invariant solutions corresponding to different symmetries. The problem is to choose the right solution from the many possible solutions. Usually physical problems accompany initial/boundary conditions as restrains to the differential equations. Many of the symmetries, if not all, may be lost due to the restrictions of the boundary conditions which leaves us the necessary subgroup of symmetries to solve the system. The boundary value problems and the symmetries corresponding to them will be addressed in Chapter 9.

Mapping a Solution into Another Solution

It is obvious that a group invariant solution cannot be mapped into another solution using the specific symmetry generator which produced that solution since the solution will map onto itself. If the starting solution is a group invariant solution corresponding to a specific generator, say X_1, then a mapping can be achieved with another generator, say X_2 with the condition that $[X_1, X_2] \neq \lambda X_1$, otherwise, the solution will still map onto itself. The algorithm is

i) Select a base generator or a linear combination of base generators which is named as mapping generator.

ii) Make sure that the mapping generator X_2 and the producing generator of the group invariant solution X_1 has a commutator relationship, $[X_1, X_2] \neq \lambda X_1$ where λ is a constant which may be zero or nonzero.

iii) Solve the Lie equations to find the Lie group of transformations corresponding to the mapping generator.

iv) Substitute the inverse transformations to the starting equation to generate another solution.

The above algorithm is valid for producing another solution from a group invariant solution. Any solution which does not belong to the class of group invariant solutions can always be mapped to other solutions via the symmetry generators. A sample case is outlined for the same PDE used in Problem 7.1.

Problem 7.2. Consider the first order partial differential equation

$$u_t = uu_x \ . \tag{7.22}$$

i) Is it possible to find another solution from the starting solution $u = -\frac{x}{t}$ using the symmetry generator $X_2 = x\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u}$ and if possible, map the solution into another solution.

ii) Repeat the calculations for $X_5 = \frac{\partial}{\partial t}$ and if possible, map the solution into another solution.

i) *Mapping via* X_2

The Lie equations (See Eqs. 3.17 and 3.18) for the transformations are

$$\frac{dt^*}{d\epsilon} = x^* \ , \ t^* = t \quad \text{at} \ \ \epsilon = 0 \, , \tag{7.23}$$

$$\frac{dx^*}{d\epsilon} = 0 \ , \ x^* = x \quad \text{at} \ \ \epsilon = 0 \, , \tag{7.24}$$

$$\frac{du^*}{d\epsilon} = u^{*2} \ , \ u^* = u \quad \text{at} \ \ \epsilon = 0 \, . \tag{7.25}$$

Solving the above equations with the initial conditions, the transformations are

$$t^* = t + \epsilon x \ , \ x^* = x \ , \ u^* = \frac{u}{1-\epsilon u} \quad . \tag{7.26}$$

To find the inverse transformations, substitute $-\epsilon$

$$t = t^* - \epsilon x^* \;,\; x = x^* \;,\; u = \frac{u^*}{1+\epsilon u^*} \quad . \tag{7.27}$$

Inserting (7.27) into the solution $u = -\frac{x}{t}$,

$$\frac{u^*}{1+\epsilon u^*} = -\frac{x^*}{t^* - \epsilon x^*} \,, \tag{7.28}$$

deleting the asterisks and solving for the dependent variable, the final mapped solution is

$$u = -\frac{x}{t} \,. \tag{7.29}$$

Although a different generator X_2 was used compared to the producing generator of $\hat{X} = X_4 + X_7$, mapping to another solution was not achieved since the commutator is zero, i.e. $[\hat{X}, X_2] = 0$.

ii) *Mapping via* X_5

The Lie equations for the transformations are

$$\frac{dt^*}{d\epsilon} = 1 \;,\; t^* = t \quad \text{at} \;\; \epsilon = 0 \,, \tag{7.30}$$

$$\frac{dx^*}{d\epsilon} = 0 \;,\; x^* = x \quad \text{at} \;\; \epsilon = 0 \,, \tag{7.31}$$

$$\frac{du^*}{d\epsilon} = 0 \;,\; u^* = u \quad \text{at} \;\; \epsilon = 0 \,. \tag{7.32}$$

Solving the above equations with the initial conditions, the transformations are

$$t^* = t + \epsilon \,,\; x^* = x \,,\; u^* = u \quad . \tag{7.33}$$

To find the inverse transformations, substitute $-\epsilon$

$$t = t^* - \epsilon \,,\; x = x^* \,,\; u = u^* \quad . \tag{7.34}$$

Inserting (7.34) into the solution $u = -\frac{x}{t}$, deleting the asterisks, the final mapped solution is

$$u = -\frac{x}{t-\epsilon} \,, \tag{7.35}$$

This time mapping has been achieved since the commutator relationship is $[\hat{X}, X_5] = -X_5 \neq \lambda \hat{X}$.

7.2. SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

The second order PDE treated in the previous chapter is retreated again. The symmetries were already calculated. Group invariant solutions as well as mapping a solution into another solution will be considered.

Group Invariant Solutions

Problem 7.3. Consider the non-dimensional fin equation with constant heat conduction coefficient and constant heat transfer coefficient

$$\theta_{xx} - N^2\theta = \theta_t \ , \tag{7.36}$$

for which the infinitesimals and generators were already calculated in Problem 6.4.

$$\xi_1 = axt + \frac{1}{2}bx + dt + e \ , \tag{7.37}$$

$$\xi_2 = at^2 + bt + c \ , \tag{7.38}$$

$$\eta = \left(-\frac{1}{4}ax^2 - \frac{1}{2}dx - \frac{1}{2}at - (at^2 + bt)N^2 + h\right)\theta + g(x,t) \ , \tag{7.39}$$

with

$$g_t = g_{xx} - N^2 g \ . \tag{7.40}$$

$$X_1 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{1}{4}x^2 + \frac{1}{2}t + t^2N^2\right)\theta\frac{\partial}{\partial \theta} \ ,$$

$$X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - N^2 t\theta\frac{\partial}{\partial \theta} \ , \ X_3 = \frac{\partial}{\partial t} \ ,$$

$$X_4 = t\frac{\partial}{\partial x} - \frac{1}{2}x\theta\frac{\partial}{\partial \theta} \ , X_5 = \frac{\partial}{\partial x} \ , X_6 = \theta\frac{\partial}{\partial \theta} \ ,$$

$$X_\infty = g(x,t)\frac{\partial}{\partial \theta} \ . \tag{7.41}$$

Calculate

i) A group invariant solution corresponding to parameter b (X_2) with all other parameters being equal to zero.

ii) A group invariant solution corresponding to parameters c and h $(cX_3 + hX_6)$ with all other parameters being equal to zero.

i) *Solution corresponding to parameter b*

For this choice $\xi_1 = \frac{1}{2}x$, $\xi_2 = t$ and $\eta = -N^2 t\theta$. The characteristic equation is,

$$\frac{dx}{\frac{1}{2}x} = \frac{dt}{t} = \frac{du}{-N^2 t\theta}\,. \tag{7.42}$$

From the first two equations,

$$\xi = \frac{x}{\sqrt{t}}\,, \tag{7.43}$$

and from the last two equations,

$$\theta = e^{-N^2 t} f(\xi)\,. \tag{7.44}$$

Hence ξ and $f(\xi)$ are the similarity variables (group invariants). Calculating the derivatives and substituting into the original PDE

$$f'' + \frac{1}{2}\xi f' = 0\,. \tag{7.45}$$

The original PDE is transformed into an ODE in terms of the similarity variables. Defining $p = f'$, the order is reduced

$$p' + \frac{1}{2}\xi p = 0\,, \tag{7.46}$$

which is a separable equation

$$\frac{dp}{p} = -\frac{1}{2}\xi d\xi\,. \tag{7.47}$$

The solution

$$p = c_1 e^{-\frac{\xi^2}{4}}\,, \tag{7.48}$$

has to be integrated to determine f

$$f = c_1 \int_0^{\xi} e^{-\frac{\xi^2}{4}} d\xi + c_2\,. \tag{7.49}$$

Substituting the above solution into (7.44) and returning back to the original variables,

$$\theta(x,t) = e^{-N^2 t}\left(c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\xi^2}{4}} d\xi + c_2\right), \tag{7.50}$$

is the group invariant solution corresponding to the generator X_2.

ii) *Solution corresponding to parameters c* and *h*

For this choice $\xi_1 = 0$, $\xi_2 = c$ and $\eta = h\theta$. The characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{c} = \frac{du}{h\theta}. \tag{7.51}$$

From the first two equations,

$$\mu = x\,, \tag{7.52}$$

and from the second and third ones,

$$\theta = e^{mt} g(\mu)\,, \tag{7.53}$$

where $m = h/c$. Calculating the derivatives and inserting into the original PDE

$$g'' - (N^2 + m)g = 0\,, \tag{7.54}$$

which is a second order constant coefficient ODE with a solution

$$g = c_1 e^{-(\sqrt{N^2+m})\mu} + c_2 e^{(\sqrt{N^2+m})\mu}\,. \tag{7.55}$$

Returning back to the original variables, the solution is

$$\theta(x,t) = e^{mt}\left(c_1 e^{-(\sqrt{N^2+m})x} + c_2 e^{(\sqrt{N^2+m})x}\right), \tag{7.56}$$

which definitely satisfies the original equation.

Mapping a Solution into Another Solution

One can more formally express the mapping of solutions corresponding to generators admitting the equation.

Theorem 7.1. If X_1 produces the group invariant solution $u = F(x,t)$, and if for another symmetry generator X_2, $[X_1, X_2] = \lambda X_1$, then the same solution is also a group invariant solution of X_2

Proof

Since the solution surface is invariant under X_1 ,

$$X_1(u - F) = 0 \; . \tag{7.57}$$

Hence

$$[X_1, X_2](u - F) = X_1 \, X_2(u - F) - X_2 \, X_1(u - F) = \lambda X_1 \, (u - F) \tag{7.58}$$

Using (7.57)

$$X_1 \, X_2(u - F) = 0 \tag{7.59}$$

which can be satisfied if $X_2(u - F) = 0$.

Problem 7.4. Consider the nonlinear fin equation

$$\theta_{xx} - N^2\theta = \theta_t \; . \tag{7.60}$$

i) Is it possible to find another solution from the starting solution

$\theta = e^{-N^2 t}\left(c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\xi^2}{4}} d\xi + c_2\right)$ corresponding to X_2 using the symmetry generator $\hat{X} = e\frac{\partial}{\partial x} + c\frac{\partial}{\partial t}$ and if possible map the solution into another solution.

ii) Is it possible to find another solution from the starting solution

$\theta(x,t) = e^{mt}\left(c_1 e^{-(\sqrt{N^2+m})x} + c_2 e^{(\sqrt{N^2+m})x}\right)$ corresponding to $cX_3 + hX_6$ using the symmetry generator $\hat{X} = e\frac{\partial}{\partial x} + c\frac{\partial}{\partial t}$ and if possible map the solution into another solution.

Solution

i) The Lie equations (See Eqs. 3.17 and 3.18) for the transformations are

$$\frac{dx^*}{d\epsilon} = e \;\; , \;\; x^* = x \quad \text{at} \;\; \epsilon = 0 \, , \tag{7.61}$$

$$\frac{dt^*}{d\epsilon} = c \ , \ t^* = t \quad \text{at} \ \ \epsilon = 0 \, , \tag{7.62}$$

$$\frac{d\theta^*}{d\epsilon} = 0 \ , \ \theta^* = \theta \quad \text{at} \ \ \epsilon = 0 \, . \tag{7.63}$$

Solving the above equations with the initial conditions, the transformations are

$$x^* = x + \epsilon e \ , \ t^* = t + \epsilon c \ , \ \theta^* = \theta \quad . \tag{7.64}$$

The inverse transformations are

$$x = x^* - \epsilon e \ , \ t = t^* - \epsilon c \ , \ \theta = \theta^* \quad . \tag{7.65}$$

Inserting (7.65) into the solution, deleting the asterisks

$$\theta = e^{-N^2(t-\epsilon c)} \left(c_1 \int_0^{\frac{x-\epsilon e}{\sqrt{t-\epsilon c}}} e^{-\frac{\xi^2}{4}} d\xi + c_2 \right), \tag{7.66}$$

is the solution mapped. The mapping has been achieved since

$\left[\hat{X}, X_2\right] = \frac{1}{2} e \frac{\partial}{\partial x} + c \frac{\partial}{\partial t} - N^2 \theta c \frac{\partial}{\partial \theta} \neq \lambda X_2.$

ii) The mapping transformations are the same,

$$x = x^* - \epsilon e \ , \ t = t^* - \epsilon c \ , \ \theta = \theta^* \ . \tag{7.67}$$

Substituting into the given solution

$$\theta(x,t) = e^{m(t-\epsilon c)} \left(c_1 e^{-(\sqrt{N^2+m})(x-\epsilon e)} + c_2 e^{(\sqrt{N^2+m})(x-\epsilon e)} \right), \tag{7.68}$$

which is merely the same solution by redefining the constants,

$$c_1^* = c_1 e^{-\epsilon mc + (\sqrt{N^2+m})\epsilon e}, \ c_2^* = c_2 e^{-\epsilon mc - (\sqrt{N^2+m})\epsilon e} \ . \tag{7.69}$$

This time mapping has not been achieved since $\left[\hat{X}, cX_3 + hX_6\right] = 0.$

7.3. HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS

The KdV equation for which the symmetries were calculated in Problem 6.5 is reconsidered. Group invariant solutions as well as mapping a solution into another solution will be discussed.

Group Invariant Solutions

Problem 7.5. Consider the KdV equation for shallow water waves

$$u_t + uu_x + u_{xxx} = 0\ , \qquad (7.70)$$

for which the infinitesimals and generators were already calculated in Problem 6.5.

$$\xi_1 = ax + bt + c\ , \qquad (7.71)$$

$$\xi_2 = 3at + d\ , \qquad (7.72)$$

$$\eta = -2au + b\ , \qquad (7.73)$$

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ \ X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}\ ,\ X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\ , \qquad (7.74)$$

Calculate

i) A group invariant solution corresponding to parameter a (X_3) with all other parameters being equal to zero.

ii) A group invariant solution corresponding to parameter b (X_4) with all other parameters being equal to zero.

i) *Solution corresponding to parameter* a (X_3)

For this choice $\xi_1 = x$, $\xi_2 = 3t$ and $\eta = -2u$. The characteristic equation is,

$$\frac{dx}{x} = \frac{dt}{3t} = \frac{du}{-2u}\ . \qquad (7.75)$$

From the first two equations,

$$\xi = \frac{x}{t^{1/3}}\ , \qquad (7.76)$$

and from the last two equations,

$$u = t^{-2/3} f(\xi)\ . \qquad (7.77)$$

Hence ξ and $f(\xi)$ are the similarity variables (group invariants). Calculating the derivatives and substituting into the original PDE and accomplishing the simplifications

$$f''' + f'\left(f - \tfrac{1}{3}\xi\right) - \tfrac{2}{3}f = 0\ . \qquad (7.78)$$

which is a nonlinear variable coefficient ODE. Once f is determined either analytically or numerically, then

$$u = t^{-2/3} f(\frac{x}{t^{1/3}}) \ , \tag{7.79}$$

is the group invariant solution.

ii) *Solution corresponding to parameter b* (X_4)

For this choice $\xi_1 = t$, $\xi_2 = 0$ and $\eta = 1$. The characteristic equation is,

$$\frac{dx}{t} = \frac{dt}{0} = \frac{du}{1} \ . \tag{7.80}$$

From the first two equations,

$$\mu = t \ , \tag{7.81}$$

and from the first and third ones,

$$u = \frac{x}{\mu} + g(\mu) \ . \tag{7.82}$$

Calculating the derivatives and inserting into the original PDE

$$g' + \frac{g}{\mu} = 0 \ , \tag{7.83}$$

with a solution

$$g = \frac{c_1}{\mu} \ . \tag{7.84}$$

Returning back to the original variables, the solution is

$$u = \frac{x}{t} + \frac{c_1}{t} \ , \tag{7.85}$$

Mapping a Solution into Another Solution

Problem 7.6. Consider the KdV equation

$$u_t + u u_x + u_{xxx} = 0 \ , \tag{7.86}$$

Is it possible to find another solution from the starting solution

$u = \frac{x}{t} + \frac{c_1}{t}$ corresponding to X_4 using the symmetry generator

$X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$ and if possible map the solution into another solution.

Solution

The Lie equations for the transformations corresponding to X_3 are

$$\frac{dx^*}{d\epsilon} = x^* \ , \ x^* = x \quad \text{at} \ \epsilon = 0 \ , \tag{7.87}$$

$$\frac{dt^*}{d\epsilon} = 3t^* \ , \ t^* = t \quad \text{at} \ \epsilon = 0 \ , \tag{7.88}$$

$$\frac{du^*}{d\epsilon} = -2u^* \ , \ u^* = u \quad \text{at} \ \epsilon = 0 \ . \tag{7.89}$$

Solving the above equations with the initial conditions, the transformations are

$$x^* = e^{\epsilon}x \ , \ t^* = e^{3\epsilon}t \ , \ u^* = e^{-2\epsilon}\,u \quad . \tag{7.90}$$

The inverse transformations are

$$x = e^{-\epsilon}x^*, \ t = e^{-3\epsilon}t^* \ , \ u = e^{2\epsilon}u^* \quad . \tag{7.91}$$

Inserting (7.91) into the solution, deleting the asterisks

$$u = \frac{x}{t} + \frac{c_1 e^{\epsilon}}{t} \tag{7.92}$$

which is the essentially the same solution if the arbitrary constant is redefined as $c_1^* = c_1 e^{\epsilon}$. Checking the commutator relation $[X_4, X_3] = -2X_4$, hence mapping into another solution cannot be achieved.

7.4. COUPLED PARTIAL DIFFERENTIAL EQUATIONS

The symmetries of the boundary layer equations of a Newtonian fluid were already calculated in Problem 6.6. A similarity transformation is given in this section.

Problem 7.7. Consider the boundary layer flow of a Newtonian fluid (Pakdemirli and Yürüsoy, 1998)

$$u_x + v_y = 0 \ , \tag{7.93}$$

$$uu_x + vu_y = U(x)U'(x) + u_{yy} \ . \tag{7.94}$$

The infinitesimals and generators were already calculated in Problem 6.6,

$$\xi_1 = ax + b \ , \tag{7.95}$$

$$\xi_2 = (a + c)y + d(x) \ , \tag{7.96}$$

$$\eta^1 = -(a + 2c)u \ , \tag{7.97}$$

$$\eta^2 = -(a + c)v + d'(x)u \ , \tag{7.98}$$

$$(ax + b)\frac{d}{dx}(UU') + (3a + 4c)UU' = 0 \quad , \tag{7.99}$$

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v} \ . \tag{7.100}$$

Find a similarity solution corresponding to the parameters $c = -a$ $(X_1 - X_3)$ with all other parameters being zero and determine the form of $U(x)$ admitting this symmetry.

Solution

The similarity transformation corresponding to the scaling symmetry (Parameter a, generator X_1) has already been given in Problem 2.8. The infinitesimals corresponding to $c = -a$ $(X_1 - X_3)$ are $\xi_1 = ax$, $\xi_2 = 0$ and $\eta^1 = au$ and $\eta^2 = 0$. The characteristic equation is,

$$\frac{dx}{x} = \frac{dy}{0} = \frac{du}{u} = \frac{dv}{0} \ . \tag{7.101}$$

The similarity variables are

$$\xi = y \ , \ u = xf(\xi) \ , \ \ v = g(\xi) \ . \tag{7.102}$$

For $b = 0$, $c = -a$, from (7.99)

$$ax\frac{d}{dx}(UU') - aUU' = 0 \ . \tag{7.103}$$

Solving the above equation,

$$UU' = k_1 x \ , \tag{7.104}$$

substituting (7.102) and (7.104) into the original system, the PDE system transforms into an ODE system

$$f + g' = 0\ , \tag{7.105}$$

$$f^2 + gf' = k_1 + f''\ . \tag{7.106}$$

The coupled system of equations can be reduced to a scalar equation by substituting $f = -g'$ into (7.106),

$$g''' - gg'' + g'^2 = k_1\ . \tag{7.107}$$

which is a nonlinear ODE to be solved with associated boundary conditions. The specific form of the outer velocity is calculated from (7.104)

$$U(x) = \sqrt{k_1 x^2 + k_2}\ . \tag{7.108}$$

Once the form of the $f(\xi)$ and $g(\xi)$ are determined either analytically or numerically, the solutions are

$$u = xf(y)\ ,\ \ v = g(y)\ . \tag{7.109}$$

The above solutions are not appropriate for the boundary conditions treated in Problem 2.8

$$u(x,0) = 0,\ v(x,0) = 0,\ u(x,\infty) = U(x)\ , \tag{7.110}$$

since the last condition does not transform appropriately under this transformation. This was not the case for the scaling symmetry (parameter a, X_1). A systematic approach for treating boundary value problems will be presented in Chapter 9.

7.5. EXERCISES

E.7.1. Consider the first order partial differential equation

$$u_t = uu_x\ ,$$

for which the infinitesimals and generators were

$$\xi_1 = (at + b)x + ct^2 + dt + e\ ,$$

$$\xi_2 = (cx + f)t + ax^2 + gx + h\ ,$$

$$\eta = -cx - f + (ax - ct + g - d)u + (at + b)u^2 ,$$

$$X_1 = xt\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + (xu + tu^2)\frac{\partial}{\partial u},\ X_2 = x\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u},$$

$$X_3 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}, X_4 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, X_5 = \frac{\partial}{\partial t},$$

$$X_6 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, X_7 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\ X_8 = \frac{\partial}{\partial x}. \tag{7.8}$$

Calculate

i) A group invariant solution corresponding to parameter b (X_2) with all other parameters being equal to zero.

ii) A group invariant solution corresponding to parameter g (X_7) with all other parameters being equal to zero.

E.7.2. For the first order PDE in Problem 7.1, can the group invariant solution corresponding to X_2 be mapped into another solution via X_7 and vice versa. Justify your answer.

E.7.3. Consider the non-dimensional fin equation with constant heat conduction coefficient and constant heat transfer coefficient

$$\theta_{xx} - N^2\theta = \theta_t ,$$

for which the infinitesimals and generators were already given

$$\xi_1 = axt + \frac{1}{2}bx + dt + e ,$$

$$\xi_2 = at^2 + bt + c ,$$

$$\eta = \left(-\frac{1}{4}ax^2 - \frac{1}{2}dx - \frac{1}{2}at - (at^2 + bt)N^2 + h\right)\theta + g(x,t) ,$$

with

$$g_t = g_{xx} - N^2 g .$$

$$X_1 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{1}{4}x^2 + \frac{1}{2}t + t^2N^2\right)\theta\frac{\partial}{\partial \theta},$$

$$X_2 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - N^2t\theta\frac{\partial}{\partial \theta},\ X_3 = \frac{\partial}{\partial t},$$

$$X_4 = t\frac{\partial}{\partial x} - \frac{1}{2}x\theta\frac{\partial}{\partial \theta}\ ,\ X_5 = \frac{\partial}{\partial x}\ ,\ X_6 = \theta\frac{\partial}{\partial \theta}\ ,$$

$$X_\infty = g(x,t)\frac{\partial}{\partial \theta}\ .$$

Calculate a group invariant solution corresponding to parameter d (X_4) with all other parameters being equal to zero.

E.7.4. For the second order PDE in Problem 7.3, can the group invariant solution corresponding to X_4 be mapped into another solution via $X_5 + X_6$ and if possible, map the X_4 solution into another solution.

E.7.5. Consider the KdV equation for shallow water waves

$$u_t + uu_x + u_{xxx} = 0$$

for which the infinitesimals and generators were given as

$$\xi_1 = ax + bt + c\ ,$$

$$\xi_2 = 3at + d\ ,$$

$$\eta = -2au + b\ ,$$

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ \ X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}\ ,\ X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\ ,$$

Calculate

i) A group invariant solution corresponding to parameter d (X_2) with all other parameters being equal to zero.

ii) A group invariant solution corresponding to parameters c and d $(cX_1 + dX_2)$ with all other parameters being equal to zero.

E.7.6. For the KdV equation in Problem 7.5, can the group invariant solution corresponding to X_4 be mapped into another solution via $cX_1 + dX_2$ and if possible, map the X_4 solution into another solution.

E.7.7. Consider the boundary layer flow of a Newtonian fluid

$$u_x + v_y = 0\ ,$$

$$uu_x + vu_y = U(x)U'(x) + u_{yy}\ .$$

The infinitesimals and generators were already calculated

$$\xi_1 = ax + b \ ,$$

$$\xi_2 = (a + c)y + d(x) \ ,$$

$$\eta^1 = -(a + 2c)u \ ,$$

$$\eta^2 = -(a + c)v + d'(x)u \ ,$$

$$(ax + b)\frac{d}{dx}(UU') + (3a + 4c)UU' = 0 \quad ,$$

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v} \ .$$

Find a similarity solution corresponding to the infinite parameter $d(x)$, i.e. X_∞ with all other parameters being zero and determine the form of $U(x)$ admitting this symmetry.