CHAPTER 6

PARTIAL DIFFERENTIAL EQUATIONS: SYMMETRY GENERATORS

In this chapter, the partial differential equations will be treated. Similar to the case of ODEs, there are two basic steps in finding solutions for partial differential equations via symmetry methods: 1) Calculate the symmetry generators 2) Use the generators to construct solutions. The first part, namely calculating the symmetries of the ODEs will be outlined in this chapter and the next chapter is devoted to constructing solutions from the generators. The recursive relations for extended infinitesimal generators will be given. Several sample problems are treated.

6.1. EXTENDED GROUPS AND INFINITESIMAL GENERATORS

A general partial differential equation of arbitrary order with one dependent variable and *n* independent variables can be written as

$$F\big(x_i, u, u_i, u_{ij}, u_{ijk}, \dots\big) = 0 \ , \ i, j, k, .. = 1,2, \dots n. \tag{6.1}$$

where x_i are the *n* independent variables and *u* is the dependent variable. For one dependent and two independent variables of $u = u(x, y)$, the variables may be

$$x_1 = x,\ x_2 = y, u_1 = u_x, u_2 = u_y, u_{11} = u_{xx}, u_{12} = u_{xy}, u_{22} = u_{yy} ,\dots \tag{6.2}$$

To express (6.1) in a more compact form, define $u_{(k)}$ as all *k*'th order derivatives. Hence, for one dependent, two independent variables, $u_{(2)}$ will represent the total of u_{11}, u_{12} and u_{22}. If Equation (6.1) is of *k*'th order, then

$$F\big(x_i, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}\big) = 0 \ . \tag{6.3}$$

In writing the Lie group of transformations for the PDE, one needs to write the transformations for the derivatives also. A sample transformation with one dependent variable and *n* independent variables will look like

$$x_i^* = x_i^*(x_j, u, \epsilon), \tag{6.4}$$

$$u^* = u^*(x_j, u, \epsilon), \tag{6.5}$$
$$u_i^* = u_i^*(x_j, u, u_{(1)}, \epsilon), \tag{6.6}$$
$$u_{i_1 i_2}^* = u_{i_1 i_2}^*(x_j, u, u_{(1)}, u_{(2)}, \epsilon), \tag{6.7}$$
$$\vdots$$
$$u_{i_1 i_2 .. i_k}^* = u_{i_1 i_2 .. i_k}^*(x_j, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}, \epsilon) , \tag{6.8}$$

where $j, i, i_1 i_2 .. i_k = 1,2, \dots n$. The transformations of higher order variables cannot be arbitrary, but dictated with the derivatives of transformations (6.4) and (6.5).

The infinitesimal transformations up to first order for the corresponding group can be written

$$x_i^* = x_i + \epsilon \xi_i(x_j, u), \tag{6.9}$$
$$u^* = u + \epsilon \eta(x_j, u), \tag{6.10}$$
$$u_i^* = u_i + \epsilon \eta_i^{(1)}(x_j, u, u_{(1)}), \tag{6.11}$$
$$u_{i_1 i_2}^* = u_{i_1 i_2} + \epsilon \eta_{i_1 i_2}^{(2)}(x_j, u, u_{(1)}, u_{(2)}), \tag{6.12}$$
$$\vdots$$
$$u_{i_1 i_2 .. i_k}^* = u_{i_1 i_2 .. i_k} + \epsilon \eta_{i_1 i_2 .. i_k}^{(k)}(x_j, u, u_{(1)}, u_{(2)}, \dots u_{(k)}) . \tag{6.13}$$

Theorem 6.1. The infinitesimals for (6.9)-(6.13) are calculated from the recursive relations

$$\eta_i^{(1)} = D_i \eta - (D_i \xi_j) u_j , \tag{6.14}$$
$$\eta_{i_1 i_2 .. i_k}^{(k)} = D_{i_k} \eta_{i_1 i_2 .. i_{k-1}}^{(k-1)} - (D_{i_k} \xi_j) u_{i_1 i_2 .. i_{k-1} j} , \tag{6.15}$$

with the total derivative being

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots + u_{i i_1 i_2 .. i_n} \frac{\partial}{\partial u_{i_1 i_2 .. i_n}} . \tag{6.16}$$

The extended infinitesimal generator corresponding to the group is therefore

$$X^{(k)} = \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + \eta_i^{(1)} \frac{\partial}{\partial u_i} + \cdots + \eta_{i_1 i_2 .. i_k}^{(k)} \frac{\partial}{\partial u_{i_1 i_2 .. i_k}} . \tag{6.17}$$

where $i = 1,2, \dots n$, $i_\ell = 1,2, \dots n$, $\ell = 1,2, \dots k$, $k = 2,3, \dots$ and summation is taken over the repeated indexes

The logic behind the derivation of the above recursive formulas is similar to the ones given for ODEs in Chapter 4 albeit with more algebra. See Bluman and Kumei (1989) for a proof of the above theorem.

The recursive relations lead to excessive terms as the number of variables and the order of equation increases which necessitates usage of symbolic computation programs. For two independent and one dependent variables $u = u(x_1, x_2)$, the extension formulas for the infinitesimals lead to

$$\eta_1^{(1)} = \frac{\partial \eta}{\partial x_1} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1}\right) u_1 - \frac{\partial \xi_2}{\partial x_1} u_2 - \frac{\partial \xi_1}{\partial u} u_1^2 - \frac{\partial \xi_2}{\partial u} u_1 u_2 \,, \tag{6.18}$$

$$\eta_2^{(1)} = \frac{\partial \eta}{\partial x_2} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi_2}{\partial x_2}\right) u_2 - \frac{\partial \xi_1}{\partial x_2} u_1 - \frac{\partial \xi_2}{\partial u} u_2^2 - \frac{\partial \xi_1}{\partial u} u_1 u_2 \,, \tag{6.19}$$

$$\begin{aligned} \eta_{11}^{(2)} = {} & \frac{\partial^2 \eta}{\partial x_1^2} + \left(2\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1^2}\right) u_1 - \frac{\partial^2 \xi_2}{\partial x_1^2} u_2 + \left(\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi_1}{\partial x_1}\right) u_{11} \\ & -2\frac{\partial \xi_2}{\partial x_1} u_{12} + \left(\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_1}{\partial x_1 \partial u}\right) u_1^2 - 2\frac{\partial^2 \xi_2}{\partial x_1 \partial u} u_1 u_2 - \frac{\partial^2 \xi_1}{\partial u^2} u_1^3 \\ & - \frac{\partial^2 \xi_2}{\partial u^2} u_1^2 u_2 - 3\frac{\partial \xi_1}{\partial u} u_1 u_{11} - \frac{\partial \xi_2}{\partial u} u_2 u_{11} - 2\frac{\partial \xi_2}{\partial u} u_1 u_{12}, \end{aligned} \tag{6.20}$$

$$\begin{aligned} \eta_{12}^{(2)} = {} & \frac{\partial^2 \eta}{\partial x_1 \partial x_2} + \left(\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_2}{\partial x_1 \partial x_2}\right) u_2 + \left(\frac{\partial^2 \eta}{\partial x_2 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1 \partial x_2}\right) u_1 \\ & - \frac{\partial \xi_2}{\partial x_1} u_{22} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1} - \frac{\partial \xi_2}{\partial x_2}\right) u_{12} - \frac{\partial \xi_1}{\partial x_2} u_{11} - \frac{\partial^2 \xi_2}{\partial x_1 \partial u} u_2^2 \\ & + \left(\frac{\partial^2 \eta}{\partial u^2} - \frac{\partial^2 \xi_1}{\partial x_1 \partial u} - \frac{\partial^2 \xi_2}{\partial x_2 \partial u}\right) u_1 u_2 - \frac{\partial^2 \xi_1}{\partial x_2 \partial u} u_1^2 - \frac{\partial^2 \xi_2}{\partial u^2} u_1 u_2^2 - \frac{\partial^2 \xi_1}{\partial u^2} u_1^2 u_2 \\ & -2\frac{\partial \xi_2}{\partial u} u_2 u_{12} - 2\frac{\partial \xi_1}{\partial u} u_1 u_{12} - \frac{\partial \xi_1}{\partial u} u_2 u_{11} - \frac{\partial \xi_2}{\partial u} u_1 u_{22} \,, \end{aligned} \tag{6.21}$$

$$\begin{aligned} \eta_{22}^{(2)} = {} & \frac{\partial^2 \eta}{\partial x_2^2} + \left(2\frac{\partial^2 \eta}{\partial x_2 \partial u} - \frac{\partial^2 \xi_2}{\partial x_2^2}\right) u_2 - \frac{\partial^2 \xi_1}{\partial x_2^2} u_1 + \left(\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi_2}{\partial x_2}\right) u_{22} \\ & -2\frac{\partial \xi_1}{\partial x_2} u_{12} + \left(\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_2}{\partial x_2 \partial u}\right) u_2^2 - 2\frac{\partial^2 \xi_1}{\partial x_2 \partial u} u_1 u_2 - \frac{\partial^2 \xi_2}{\partial u^2} u_2^3 \\ & - \frac{\partial^2 \xi_1}{\partial u^2} u_1 u_2^2 - 3\frac{\partial \xi_2}{\partial u} u_2 u_{22} - \frac{\partial \xi_1}{\partial u} u_1 u_{22} - 2\frac{\partial \xi_1}{\partial u} u_2 u_{12} \,. \end{aligned} \tag{6.22}$$

Note that $\eta_{12}^{(1)} = \eta_{21}^{(1)}$ from the symmetry of the derivative operator and no need to be calculated.

In the case of systems of differential equations, the number of dependent variables increases as the number of coupled equations increases. Assume m dependent variables $u^\alpha, \alpha = 1,2, \dots m$, with n independent variables, the recursive relations in Theorem (6.1) assumes the modified form (Bluman and Kumei, 1989)

$$\eta_i^{(1)\mu} = D_i\eta^\mu - (D_i\xi_j)u_j^\mu \ , \tag{6.23}$$

$$\eta_{i_1 i_2 .. i_k}^{(k)\mu} = D_{i_k}\eta_{i_1 i_2 .. i_{k-1}}^{(k-1)\mu} - (D_{i_k}\xi_j)u_{i_1 i_2 .. i_{k-1} j}^\mu \ , \tag{6.24}$$

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_j^\mu} + \cdots + u_{i i_1 i_2 .. i_k}^\mu \frac{\partial}{\partial u_{i_1 i_2 .. i_k}^\mu} \ , \tag{6.25}$$

$$X^{(k)} = \xi_i \frac{\partial}{\partial x_i} + \eta^\mu \frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu} \frac{\partial}{\partial u_i^\mu} + \cdots + \eta_{i_1 i_2 .. i_k}^{(k)\mu} \frac{\partial}{\partial u_{i_1 i_2 .. i_k}^\mu} \ , \tag{6.26}$$

where $i = 1,2, \dots n$, $i_\ell = 1,2, \dots n$, $\ell = 1,2, \dots k$, $k = 2,3, \dots$ $\mu = 1,2, \dots m$ and summation is taken over all repeated indexes.

Problem 6.1. Determine the second order extended infinitesimals and generators for the three parameter translational group with $u = u(x, y)$

$$x^* = x + \epsilon a \ , \tag{6.27}$$

$$y^* = y + \epsilon b \ , \tag{6.28}$$

$$u^* = u + \epsilon c \ . \tag{6.29}$$

Solution

Define $x_1 = x$ and $x_2 = y$. Since the infinitesimals are all constants, i.e., $\xi_1 = a$, $\xi_2 = b$ and $\eta = c$, all derivatives vanish in Equations (6.18)-(6.22). Hence, $\eta_i^{(1)} = 0$ and $\eta_{ij}^{(2)} = 0$. The extended generator to second order is $X^{(2)} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial u}$.

Problem 6.2. Determine the second order extended infinitesimals and generators for the three parameter scaling group with $u = u(x, y)$

$$x^* = x + \epsilon a x \ , \tag{6.30}$$

$$y^* = y + \epsilon b y \ , \tag{6.31}$$

$$u^* = u + \epsilon c u \ . \tag{6.32}$$

Solution

Define $x_1 = x$ and $x_2 = y$. The infinitesimals are $\xi_1 = ax$, $\xi_2 = by$ and $\eta = cu$. Inserting into (6.18)-(6.22)

$$\eta_1^{(1)} = (c-a)u_1,\ \eta_2^{(1)} = (c-b)u_2\ , \tag{6.33}$$

$$\eta_{11}^{(1)} = (c-2a)u_{11}\ , \eta_{12}^{(1)} = (c-a-b)u_{12}\ , \eta_{22}^{(1)} = (c-2b)u_{22}\ . \tag{6.34}$$

The second order extended generator is

$$X^{(2)} = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cu\frac{\partial}{\partial u} + (c-a)u_1\frac{\partial}{\partial u_1} + (c-b)u_2\frac{\partial}{\partial u_2}$$

$$+(c-2a)u_{11}\frac{\partial}{\partial u_{11}} + (c-a-b)u_{12}\frac{\partial}{\partial u_{12}} + (c-2b)u_{22}\frac{\partial}{\partial u_{22}}\,. \tag{6.35}$$

6.2. INVARIANCE OF A PARTIAL DIFFERENTIAL EQUATION

The invariance definition and theorem are much similar to the case of ODEs given in Chapter 4. The definition and invariance theorem is given for one dependent and *n* independent variables.

Definition 6.1. The partial differential equation of k'th order $F(x_i, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0$ is invariant under the extended Lie group of transformations (6.9)-(6.13) if and only if

$$F(x_i^*, u^*, u_{(1)}^*, u_{(2)}^*, \dots\ u_{(k)}^*) = 0 \text{ when } F(x_i, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0 \tag{6.36}$$

The relevant theorem for calculating the symmetries immediately follow from the definition:

Theorem 6.2. If the partial differential equation of k'th order $F(x_i, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0$ is invariant under the given Lie Group of transformations (6.9)-(6.13), then

$$X^{(k)}F \equiv 0 \text{ when } F = 0\ \ , \tag{6.37}$$

where $X^{(k)}$ is the k'th order extended generator of the group admitted by the equation

The proof has the essential steps of the proof of Theorem 3.1 given for invariance of algebraic equations and therefore skipped.

6.3. FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

In the case of first order ordinary differential equations, it is shown in Chapter 4 that the equations are not solvable in general since there are two unknowns with a single determining equation for the infinitesimals. In the case of partial differential equations, the determining equation for infinitesimals can indeed be separated increasing the possibility of obtaining a solution. However, as is outlined in the following example problem, one may still need some simplifying assumptions to determine the infinitesimals which make the Lie Algebra a subalgebra of the complete group.

Problem 6.3. Consider the first order partial differential equation

$$u_t = uu_x \ . \tag{6.38}$$

Calculate the infinitesimals and the base generators admitting the equation.

Solution

Defining $x_1 = t$, $x_2 = x$, $u_1 = u_t$ and $u_2 = u_x$, the equation is

$$F(u, u_1, u_2) = u_1 - uu_2 = 0 \ . \tag{6.39}$$

The infinitesimal generator extended to first order is

$$X^{(1)} = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u} + \eta_1^{(1)} \frac{\partial}{\partial u_1} + \eta_2^{(1)} \frac{\partial}{\partial u_2} \ . \tag{6.40}$$

The invariance condition $X^{(1)}F \equiv 0$ requires

$$-\eta u_2 + \eta_1^{(1)} - \eta_2^{(1)} u = 0 \ . \tag{6.41}$$

Substituting from (6.18) and (6.19) the extended infinitesimals and using $u_1 = uu_2$ wherever u_1 appears, the determining block can be considered as a polynomial in terms of u_2 which separates into two equations when the coefficients of the polynomials with respect to higher order variables are equated to zero

$$\frac{\partial \eta}{\partial x_1} - u \frac{\partial \eta}{\partial x_2} = 0 \ , \tag{6.42}$$

$$-\eta - \frac{\partial \xi_1}{\partial x_1} u - \frac{\partial \xi_2}{\partial x_1} + \frac{\partial \xi_2}{\partial x_2} u + \frac{\partial \xi_1}{\partial x_2} u^2 = 0 . \tag{6.43}$$

Solving η,

$$\eta = -\frac{\partial \xi_2}{\partial x_1} + \left(\frac{\partial \xi_2}{\partial x_2} - \frac{\partial \xi_1}{\partial x_1}\right) u + \frac{\partial \xi_1}{\partial x_2} u^2 , \tag{6.44}$$

and substituting into (6.42) yields

$$-\frac{\partial^2 \xi_2}{\partial x_1^2} + \left(2 \frac{\partial^2 \xi_2}{\partial x_1 \partial x_2} - \frac{\partial^2 \xi_1}{\partial x_1^2}\right) u + \left(2 \frac{\partial^2 \xi_1}{\partial x_1 \partial x_2} - \frac{\partial^2 \xi_2}{\partial x_2^2}\right) u^2 - \frac{\partial^2 \xi_1}{\partial x_2^2} u^3 = 0 . \tag{6.45}$$

Since ξ_1 and ξ_2 both depend on u, the equation is inseparable with respect to u in general. The two unknowns with one equation cannot be solved. However, one may make the simplifying assumption that $\xi_1 = \xi_1(x_1, x_2)$ and $\xi_2 = \xi_2(x_1, x_2)$ which leads to a separation with respect to u

$$\frac{\partial^2 \xi_2}{\partial x_1^2} = 0 , \tag{6.46}$$

$$2 \frac{\partial^2 \xi_2}{\partial x_1 \partial x_2} - \frac{\partial^2 \xi_1}{\partial x_1^2} = 0 , \tag{6.47}$$

$$2 \frac{\partial^2 \xi_1}{\partial x_1 \partial x_2} - \frac{\partial^2 \xi_2}{\partial x_2^2} = 0 , \tag{6.48}$$

$$\frac{\partial^2 \xi_1}{\partial x_2^2} = 0 . \tag{6.49}$$

From (6.49)

$$\xi_1 = a(x_1) x_2 + b(x_1) . \tag{6.50}$$

From (6.46)

$$\xi_2 = c(x_2) x_1 + d(x_2) . \tag{6.51}$$

Substituting ξ_1 and ξ_2 into (6.47) and solving

$$c'(x_2) = \frac{1}{2} a''(x_1) x_2 + \frac{1}{2} b''(x_1) , \tag{6.52}$$

which is a contradiction since the left hand side depends solely on x_2 whereas the right hand side depends on both x_1 and x_2. The contradiction is removed if

$a''(x_1) = 2a_1$ and $b''(x_1) = 2b_1$ for some constants a_1 and b_1. Integrating them

$$a(x_1) = a_1x_1^2 + a_2x_1 + a_3 \,, \tag{6.53}$$

$$b(x_1) = b_1x_1^2 + b_2x_1 + b_3 \,. \tag{6.54}$$

Integrating (6.52) in view of (6.53) and (6.54) yields

$$c(x_2) = \frac{1}{2}a_1x_2^2 + b_1x_2 + c_1. \tag{6.55}$$

The last equation to be satisfied is (6.48). Substituting (6.50) and (6.51) with (6.53)-(6.55) and solving

$$d''(x_2) = 3a_1x_1 + 2a_2 \,. \tag{6.56}$$

Since the left hand side is a function of x_2 only, the contradiction in the above equation can be removed if

$$a_1 = 0. \tag{6.57}$$

Hence

$$d(x_2) = a_2x_2^2 + d_2x_2 + d_3 \,. \tag{6.58}$$

Substituting all into (6.50), (6.51) and (6.44), the infinitesimals are

$$\xi_1 = (a_2x_1 + a_3)x_2 + b_1x_1^2 + b_2x_1 + b_3 \,, \tag{6.59}$$

$$\xi_2 = (b_1x_2 + c_1)x_1 + a_2x_2^2 + d_2x_2 + d_3 \,, \tag{6.60}$$

$$\eta = -b_1x_2 - c_1 + (a_2x_2 - b_1x_1 + d_2 - b_2)u + (a_2x_1 + a_3)u^2 \,. \tag{6.61}$$

Defining $a_2 = a, a_3 = b, b_1 = c, b_2 = d, b_3 = e, c_1 = f,\ d_2 = g, d_3 = h$ and bearing in mind that $x_1 = t$ and $x_2 = x$, the infinitesimals are

$$\xi_1 = (at + b)x + ct^2 + dt + e \,, \tag{6.62}$$

$$\xi_2 = (cx + f)t + ax^2 + gx + h \,, \tag{6.63}$$

$$\eta = -cx - f + (ax - ct + g - d)u + (at + b)u^2 \,, \tag{6.64}$$

which constitutes a 8-parameter finite Lie group of transformations with the base generators

$$X_1 = xt\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + (xu + tu^2)\frac{\partial}{\partial u}\,,\;\; X_2 = x\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u}\,,$$

$$X_3 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}\,, X_4 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}\,, X_5 = \frac{\partial}{\partial t}\,,$$

$$X_6 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}\,, X_7 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\,,\;\; X_8 = \frac{\partial}{\partial x}\,. \tag{6.65}$$

Among the generators, X_4 and X_7 are scaling, X_5 and X_8 are translational symmetries. X_6 and the first three generators correspond to more complex symmetries. Note that due to the simplifying assumption after equation (6.45), the above symmetries do not constitute all the symmetries of the equation.

6.4. SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

In the case of second order partial differential equations, similar to the case of ODEs, the determining equations are in general solvable since the partial differential equation system is over-determined after separation into higher order variables. A sample problem from heat transfer is treated.

Problem 6.4. Consider the non-dimensional fin equation with constant heat conduction coefficient and constant heat transfer coefficient (Pakdemirli and Şahin, 2004)

$$\theta_{xx} - N^2\theta = \theta_t\,, \tag{6.66}$$

where θ is the dimensionless temperature, x and t are the spatial and time variables. Calculate the infinitesimals and the base generators admitting the equation.

Solution

Defining $x_1 = x$, $x_2 = t, u = \theta$, $u_1 = \theta_x$, $u_2 = \theta_t$ and $u_{11} = \theta_{xx}$, the equation is

$$u_2 = u_{11} - N^2u\,. \tag{6.67}$$

The infinitesimal generator extended to second order is

$$X^{(2)} = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + \eta\frac{\partial}{\partial u} + \eta_1^{(1)}\frac{\partial}{\partial u_1} + \eta_2^{(1)}\frac{\partial}{\partial u_2} + \eta_{11}^{(2)}\frac{\partial}{\partial u_{11}}\,. \tag{6.68}$$

The invariance condition requires

$$\eta_2^{(1)} = \eta_{11}^{(2)} - N^2\eta \, . \tag{6.69}$$

Substituting from (6.19) and (6.20) the extended infinitesimals and using $u_{11} = u_2 + N^2 u$ wherever u_{11} appears, the determining block can be considered as a polynomial in terms of higher order variables which separates into the following equations when the coefficients of the polynomials are equated to zero

$$\frac{\partial \eta}{\partial x_2} = \frac{\partial^2 \eta}{\partial x_1^2} + \left(\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi_1}{\partial x_1}\right) N^2 u - N^2 \eta \, , \tag{6.70}$$

$$\frac{\partial \xi_2}{\partial x_2} = \frac{\partial^2 \xi_2}{\partial x_1^2} + 2\frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_2}{\partial u} N^2 u \, , \tag{6.71}$$

$$-\frac{\partial \xi_1}{\partial x_2} = 2\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1^2} - 3\frac{\partial \xi_1}{\partial u} N^2 u \, , \tag{6.72}$$

$$\frac{\partial \xi_1}{\partial u} = -\frac{\partial^2 \xi_2}{\partial x_1 \partial u} \, , \tag{6.73}$$

$$0 = \frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi_2}{\partial x_1 \partial u} \, , \tag{6.74}$$

$$0 = \frac{\partial^2 \xi_1}{\partial u^2} \, , \tag{6.75}$$

$$0 = \frac{\partial^2 \xi_2}{\partial u^2} \, , \tag{6.76}$$

$$0 = \frac{\partial \xi_2}{\partial x_1} \, , \tag{6.77}$$

$$0 = \frac{\partial \xi_2}{\partial u} \, . \tag{6.78}$$

From (6.77) and (6.78) $\xi_2 = \xi_2(x_2)$. From (6.73) then $\xi_1 = \xi_1(x_1, x_2)$. The remaining equations are

$$\frac{\partial \eta}{\partial x_2} = \frac{\partial^2 \eta}{\partial x_1^2} + \left(\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi_1}{\partial x_1}\right) N^2 u - N^2 \eta \, , \tag{6.79}$$

$$\frac{d \xi_2}{d x_2} = 2\frac{\partial \xi_1}{\partial x_1} \, , \tag{6.80}$$

$$-\frac{\partial \xi_1}{\partial x_2} = 2\frac{\partial^2 \eta}{\partial x_1 \partial u} - \frac{\partial^2 \xi_1}{\partial x_1^2} \, , \tag{6.81}$$

$$0 = \frac{\partial^2 \eta}{\partial u^2} \, . \tag{6.82}$$

From (6.80)

$$\xi_1 = \frac{1}{2}\xi_2'(x_2)x_1 + a(x_2) \, . \tag{6.83}$$

From (6.82)

$$\eta = b(x_1, x_2)u + g(x_1, x_2) \, . \tag{6.84}$$

Substituting ξ_1 and η into (6.81)

$$-\frac{1}{2}\xi_2''(x_2)x_1 - a'(x_2) = 2\frac{\partial b}{\partial x_1} \, , \tag{6.85}$$

and solving for b

$$b = -\frac{1}{8}\xi_2''(x_2)x_1^2 - \frac{1}{2}a'(x_2)x_1 + d(x_2) \, . \tag{6.86}$$

Substitute all into the last equation, namely (6.79) and separate with respect to u

$$\frac{\partial g}{\partial x_2} = \frac{\partial^2 g}{\partial x_1^2} - N^2 g \, , \tag{6.87}$$

$$\frac{\partial b}{\partial x_2} = \frac{\partial^2 b}{\partial x_1^2} - \xi_2'(x_2)N^2 \, . \tag{6.88}$$

Substituting b from (6.86) into (6.88) yields

$$-\frac{1}{8}\xi_2'''(x_2)x_1^2 - \frac{1}{2}a''(x_2)x_1 + d'(x_2) = -\frac{1}{4}\xi_2''(x_2) - \xi_2'(x_2)N^2 \, , \tag{6.89}$$

which separates further with respect to the variable x_1

$$-\frac{1}{8}\xi_2'''(x_2) = 0 \, , \tag{6.90}$$

$$-\frac{1}{2}a''(x_2) = 0 \, , \tag{6.91}$$

$$d'(x_2) = -\frac{1}{4}\xi_2''(x_2) - \xi_2'(x_2)N^2 \, . \tag{6.92}$$

Solving the above equations

$$\xi_2(x_2) = k_1 x_2^2 + k_2 x_2 + k_3 \, , \tag{6.93}$$

$$a(x_2) = a_1 x_2 + a_2 \ , \tag{6.94}$$

$$d(x_2) = -\tfrac{1}{2} k_1 x_2 - (k_1 x_2^2 + k_2 x_2) N^2 + k_4 \ . \tag{6.95}$$

Substituting (6.93) and (6.94) into (6.83)

$$\xi_1 = k_1 x_1 x_2 + \tfrac{1}{2} k_2 x_1 + a_1 x_2 + a_2 \ . \tag{6.96}$$

From (6.84), using (6.86), (6.93), (6.94) and (6.95)

$$\eta = \left(-\tfrac{1}{4} k_1 x_1^2 - \tfrac{1}{2} a_1 x_1 - \tfrac{1}{2} k_1 x_2 - (k_1 x_2^2 + k_2 x_2) N^2 + k_4\right) u + g(x_1, x_2) \ . \tag{6.97}$$

Defining new parameters $a = k_1$, $b = k_2$, $c = k_3$, $d = a_1$, $e = a_2$, $h = k_4$ and bearing in mind that $x = x_1$, $t = x_2$ and $\theta = u$ the infinitesimals are

$$\xi_1 = axt + \tfrac{1}{2} bx + dt + e \ , \tag{6.98}$$

$$\xi_2 = at^2 + bt + c \ , \tag{6.99}$$

$$\eta = \left(-\tfrac{1}{4} ax^2 - \tfrac{1}{2} dx - \tfrac{1}{2} at - (at^2 + bt) N^2 + h\right) \theta + g(x,t) \ , \tag{6.100}$$

with

$$g_t = g_{xx} - N^2 g \ . \tag{6.101}$$

There are 6 finite parameter Lie group of transformations (*a*,*b*,*c*,*d*,*e*,*h*) and one infinite parameter Lie group of transformation $g(x,t)$. Note that $g(x,t)$ is indeed a solution of the original equation. For linear equations, a function satisfying the original equation does always appear as an infinite parameter group.

The base generators are

$$X_1 = xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} - \left(\tfrac{1}{4} x^2 + \tfrac{1}{2} t + t^2 N^2\right) \theta \frac{\partial}{\partial \theta} \ ,$$

$$X_2 = \tfrac{1}{2} x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - N^2 t \theta \frac{\partial}{\partial \theta} \ , \ X_3 = \frac{\partial}{\partial t} \ ,$$

$$X_4 = t \frac{\partial}{\partial x} - \tfrac{1}{2} x \theta \frac{\partial}{\partial \theta} \ , X_5 = \frac{\partial}{\partial x} \ , X_6 = \theta \frac{\partial}{\partial \theta} \ ,$$

$$X_\infty = g(x,t) \frac{\partial}{\partial \theta} \ . \tag{6.102}$$

Contrary to the first order case, in the case of second order partial differential equations, the whole group is retrieved. Results are compatible with Pakdemirli and Şahin (2004) for constant heat conduction and heat transfer coefficients.

6.5. HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS

For higher order partial differential equations, the over-determined system appearing after the invariance conditions is a large scale system of equations which can be solvable. However, as the order of the equation increases, the algebra increases extensively which may require symbolic computational programs (Baumann, 2000). A third order well-known problem is treated.

Problem 6.5. Consider the non-dimensional Korteweg de Vries equation (KdV) (Baumann, 2000)

$$u_t + uu_x + u_{xxx} = 0 \,, \tag{6.103}$$

which is a well-known mathematical formulation to model the wave motion in shallow waters. Calculate the infinitesimals and the base generators admitting the equation.

Solution

Defining $x_1 = x$, $x_2 = t$, $u_1 = u_x$, $u_2 = u_t$ and $u_{111} = u_{xxx}$, the equation is

$$u_2 + uu_1 + u_{111} = 0 \,. \tag{6.104}$$

The infinitesimal generator extended to third order is

$$X^{(2)} = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u} + \eta_1^{(1)} \frac{\partial}{\partial u_1} + \eta_2^{(1)} \frac{\partial}{\partial u_2} + \eta_{111}^{(3)} \frac{\partial}{\partial u_{111}} \,. \tag{6.105}$$

The invariance condition requires

$$\eta u_1 + u\eta_1^{(1)} + \eta_2^{(1)} + \eta_{111}^{(3)} = 0 \,. \tag{6.106}$$

From (6.15) and (6.16),

$$\eta_{111}^{(3)} = D_1 \eta_{11}^{(2)} - (D_1 \xi_1) u_{111} - (D_1 \xi_2) u_{112} \,, \tag{6.107}$$

$$D_1 = \frac{\partial}{\partial x_1} + u_1 \frac{\partial}{\partial u} + u_{11} \frac{\partial}{\partial u_{11}} + u_{12} \frac{\partial}{\partial u_2} + u_{111} \frac{\partial}{\partial u_{11}} + u_{112} \frac{\partial}{\partial u_{12}} + u_{122} \frac{\partial}{\partial u_{22}} \,. \tag{6.108}$$

where $\eta_{11}^{(2)}$ is given in (6.20) and the first extensions $\eta_1^{(1)}$ and $\eta_2^{(1)}$ in (6.18) and (6.19). The invariance condition can now be calculated and $u_{111} = -u_2 - uu_1$ has to be submitted wherever u_{111} appears. The block of equations can be separated with respect to higher order variables. The coefficient of $u_1 u_{112}$ requires $\frac{\partial \xi_2}{\partial u} = 0$, the coefficient of u_{112} requires $\frac{\partial \xi_2}{\partial x_1} = 0$ and the coefficient of u_{11}^2 requires $\frac{\partial \xi_1}{\partial u} = 0$. Hence

$$\xi_1 = \xi_1(x_1, x_2), \;\; \xi_2 = \xi_2(x_2) \,, \tag{6.109}$$

which simplifies the determining equations,

$$u\frac{\partial \eta}{\partial x_1} + \frac{\partial \eta}{\partial x_2} + \frac{\partial^3 \eta}{\partial x_1^3} = 0 \,, \tag{6.110}$$

$$\eta - \frac{\partial \xi_1}{\partial x_2} + 3\frac{\partial^3 \eta}{\partial x_1^2 \partial u} - \frac{\partial^3 \xi_1}{\partial x_1^3} + 2u\frac{\partial \xi_1}{\partial x_1} = 0 \,, \tag{6.111}$$

$$-\frac{\partial \xi_2}{\partial x_2} + 3\frac{\partial \xi_1}{\partial x_1} = 0 \,, \tag{6.112}$$

$$-\frac{\partial^2 \xi_1}{\partial x_1^2} + \frac{\partial^2 \eta}{\partial x_1 \partial u} = 0 \,, \tag{6.113}$$

$$\frac{\partial^3 \eta}{\partial x_1 \partial u^2} = 0 \,, \tag{6.114}$$

$$\frac{\partial^2 \eta}{\partial u^2} = 0 \,, \tag{6.115}$$

$$\frac{\partial^3 \eta}{\partial u^3} = 0 \,. \tag{6.116}$$

From (6.115)

$$\eta = a(x_1, x_2)u + b(x_1, x_2) \,, \tag{6.117}$$

which satisfies (6.116) and (6.114). From (6.112)

$$\xi_1 = \frac{1}{3}\xi_2'(x_2)x_1 + c(x_2) \,. \tag{6.118}$$

From (6.113), $\frac{\partial a}{\partial x_1} = 0$, hence $a = a(x_2)$. Then

$$\eta = a(x_2)u + b(x_1, x_2) \,. \tag{6.119}$$

Substituting ξ_1 and η into (6.111) and separating with respect to the variable u yields

$$a(x_2) + \frac{2}{3}\xi_2'(x_2) = 0 \; , \tag{6.120}$$

$$b(x_1, x_2) - \frac{1}{3}\xi_2''(x_2)x_1 - c'(x_2) = 0 \; . \tag{6.121}$$

Solving $a(x_2)$ and $b(x_1, x_2)$ from above and substituting into (6.119)

$$\eta = -\frac{2}{3}\xi_2'(x_2)u + \frac{1}{3}\xi_2''(x_2)x_1 + c'(x_2) \; . \tag{6.122}$$

The last equation to be satisfied is (6.110) which separate with respect to u

$$-\frac{1}{3}\xi_2''(x_2) = 0 \; , \tag{6.123}$$

$$\frac{1}{3}\xi_2'''(x_2)x_1 + c''(x_2) = 0 \; , \tag{6.124}$$

with solutions

$$\xi_2(x_2) = a_1 x_2 + a_2 \; , \tag{6.125}$$

$$c(x_2) = c_1 x_2 + c_2 \; . \tag{6.126}$$

Substituting the above results into (6.122) and (6.118), the infinitesimals are finally determined

$$\xi_1 = \frac{1}{3}a_1 x_1 + c_1 x_2 + c_2 \; , \tag{6.127}$$

$$\xi_2 = a_1 x_2 + a_2 \; , \tag{6.128}$$

$$\eta = -\frac{2}{3}a_1 u + c_1 \; . \tag{6.129}$$

Defining new parameters $a_1 = 3a$, $c_1 = b$, $c_2 = c$, $a_2 = d$ and bearing in mind that $x_1 = x$ and $x_2 = t$, the infinitesimals are

$$\xi_1 = ax + bt + c \; , \tag{6.130}$$

$$\xi_2 = 3at + d \; , \tag{6.131}$$

$$\eta = -2au + b \; . \tag{6.132}$$

There are 4 finite parameter Lie group of transformations. The base generators are

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ \ X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u},\ X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad (6.133)$$

which are exactly the same generators calculated by Bauman (2000) using Mathematica program.

6.6. COUPLED PARTIAL DIFFERENTIAL EQUATIONS

For coupled equations, there is more than one dependent variable. A sample problem from fluid mechanics in which there are two independent and two dependent variables will be solved. As mentioned earlier, as the number of variables increases, the number of terms in the infinitesimals increase.

Problem 6.6. Consider the non-dimensional steady state boundary layer flow of a Newtonian fluid modeled by the coupled equations (Pakdemirli and Yürüsoy, 1998)

$$u_x + v_y = 0\,, \quad (6.134)$$

$$uu_x + vu_y = U(x)U'(x) + u_{yy}\,, \quad (6.135)$$

where *x* is the coordinate parallel to the boundary and *y* is the coordinate vertical to it. $u(x,y)$ and $v(x,y)$ are the *x* and *y* components of velocity inside the boundary layer respectively. $U(x)$ is the inviscid velocity component in *x* direction outside the boundary layer. Calculate the infinitesimals and the base generators admitting the equation.

Solution

Defining $x_1 = x$, $x_2 = y$, $u^1 = u$, $u^2 = v$, $u_1^1 = u_x$, $u_2^1 = u_y$, $u_1^2 = v_x$, $u_2^2 = v_y$ and $u_{22}^1 = u_{yy}$, the equations are

$$u_1^1 + u_2^2 = 0\,, \quad (6.136)$$

$$u^1u_1^1 + u^2u_2^1 = U(x_1)U'(x_1) + u_{22}^1\,. \quad (6.137)$$

The infinitesimal generator extended to second order is

$$X^{(2)} = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + \eta^1\frac{\partial}{\partial u^1} + \eta^2\frac{\partial}{\partial u^2} + \eta_1^{(1)1}\frac{\partial}{\partial u_1^1} + \eta_2^{(1)1}\frac{\partial}{\partial u_2^1}$$

$$+\eta_2^{(1)2}\frac{\partial}{\partial u_2^2}+\eta_{22}^{(2)1}\frac{\partial}{\partial u_{22}^1} \quad . \tag{6.138}$$

The first order extended infinitesimals are calculated for two dependent and independent variables from the recursion relation given in (6.23)

$$\eta_1^{(1)1}=D_1\eta^1-\left(D_1\xi_j\right)u_j^1=D_1\eta^1-(D_1\xi_1)u_1^1-(D_1\xi_2)u_2^1\,, \tag{6.139}$$

where

$$D_1=\frac{\partial}{\partial x_1}+u_1^1\frac{\partial}{\partial u^1}+u_1^2\frac{\partial}{\partial u^2}\,. \tag{6.140}$$

Hence

$$\eta_1^{(1)1}=\frac{\partial\eta^1}{\partial x_1}+u_1^1\left(\frac{\partial\eta^1}{\partial u^1}-\frac{\partial\xi_1}{\partial x_1}\right)+u_1^2\frac{\partial\eta^1}{\partial u^2}-u_2^1\frac{\partial\xi_2}{\partial x_1}-(u_1^1)^2\frac{\partial\xi_1}{\partial u^1}$$
$$-u_1^1u_1^2\frac{\partial\xi_1}{\partial u^2}-u_1^1u_2^1\frac{\partial\xi_2}{\partial u^1}-u_2^1u_1^2\frac{\partial\xi_2}{\partial u^2}\,. \tag{6.141}$$

Similarly

$$\eta_2^{(1)1}=\frac{\partial\eta^1}{\partial x_2}-u_1^1\frac{\partial\xi_1}{\partial x_2}+u_2^1\left(\frac{\partial\eta^1}{\partial u^1}-\frac{\partial\xi_2}{\partial x_2}\right)+u_2^2\frac{\partial\eta^1}{\partial u^2}-u_1^1u_2^1\frac{\partial\xi_1}{\partial u^1}$$
$$-u_1^1u_2^2\frac{\partial\xi_1}{\partial u^2}-(u_2^1)^2\frac{\partial\xi_2}{\partial u^1}-u_2^1u_2^2\frac{\partial\xi_2}{\partial u^2}\,, \tag{6.142}$$

$$\eta_2^{(1)2}=\frac{\partial\eta^2}{\partial x_2}+u_2^1\frac{\partial\eta^2}{\partial u^1}-u_1^2\frac{\partial\xi_1}{\partial x_2}+u_2^2\left(\frac{\partial\eta^2}{\partial u^2}-\frac{\partial\xi_2}{\partial x_2}\right)-u_2^1u_1^2\frac{\partial\xi_1}{\partial u^1}$$
$$-u_1^2u_2^2\frac{\partial\xi_1}{\partial u^2}-u_2^1u_2^2\frac{\partial\xi_2}{\partial u^1}-(u_2^2)^2\frac{\partial\xi_2}{\partial u^2}\,. \tag{6.143}$$

For second extensions, only the term $\eta_{22}^{(2)1}$ is needed. From (6.24)

$$\eta_{22}^{(2)1}=D_2\eta_2^{(1)1}-\left(D_2\xi_j\right)u_{2j}^1=D_2\eta_2^{(1)1}-(D_2\xi_1)u_{21}^1-(D_2\xi_2)u_{22}^1\,, \tag{6.144}$$

where

$$D_2=\frac{\partial}{\partial x_2}+u_2^1\frac{\partial}{\partial u^1}+u_2^2\frac{\partial}{\partial u^2}+u_{21}^1\frac{\partial}{\partial u_1^1}+u_{22}^1\frac{\partial}{\partial u_2^1}+u_{21}^2\frac{\partial}{\partial u_1^2}+u_{22}^2\frac{\partial}{\partial u_2^2}\,. \tag{6.145}$$

A straightforward calculation yields

$$\eta_{22}^{(2)1}=\frac{\partial^2\eta^1}{\partial x_2^2}-u_1^1\frac{\partial^2\xi_1}{\partial x_2^2}+u_2^1\left(2\frac{\partial^2\eta^1}{\partial x_2\partial u^1}-\frac{\partial^2\xi_2}{\partial x_2^2}\right)+2u_2^2\frac{\partial^2\eta^1}{\partial x_2\partial u^2}$$
$$-u_1^1u_2^1\left(\frac{\partial^2\xi_1}{\partial x_2\partial u^1}+\frac{\partial^2\xi_2}{\partial x_2\partial u^1}\right)-2u_1^1u_2^2\frac{\partial^2\xi_1}{\partial x_2\partial u^2}+(u_2^1)^2\left(\frac{\partial^2\eta^1}{\partial(u^1)^2}-2\frac{\partial^2\xi_2}{\partial x_2\partial u^1}\right)$$

$$+2u_2^1u_2^2\left(\frac{\partial^2\eta^1}{\partial u^1\partial u^2}-\frac{\partial^2\xi_2}{\partial x_2\partial u^2}\right)+(u_2^2)^2\frac{\partial^2\eta^1}{\partial(u^2)^2}-u_1^1(u_2^1)^2\frac{\partial^2\xi_1}{\partial(u^1)^2}$$
$$-2u_1^1u_2^1u_2^2\frac{\partial^2\xi_1}{\partial u^1\partial u^2}-(u_2^1)^3\frac{\partial^2\xi_2}{\partial(u^1)^2}-2(u_2^1)^2u_2^2\frac{\partial^2\xi_2}{\partial u^1\partial u^2}-u_1^1(u_2^2)^2\frac{\partial^2\xi_1}{\partial(u^2)^2}$$
$$-u_2^1(u_2^2)^2\frac{\partial^2\xi_2}{\partial(u^2)^2}-2u_{21}^1\frac{\partial\xi_1}{\partial x_2}+u_{22}^1\left(\frac{\partial\eta^1}{\partial u^1}-2\frac{\partial\xi_2}{\partial x_2}\right)+u_{22}^2\frac{\partial\eta^1}{\partial u^2}$$
$$-2u_2^1u_{21}^1\frac{\partial\xi_1}{\partial u^1}-2u_2^2u_{21}^1\frac{\partial\xi_1}{\partial u^2}-u_1^1u_{22}^1\frac{\partial\xi_1}{\partial u^1}-3u_2^1u_{22}^1\frac{\partial\xi_2}{\partial u^1}$$
$$-2u_2^2u_{22}^1\frac{\partial\xi_2}{\partial u^2}-u_1^1u_{22}^2\frac{\partial\xi_1}{\partial u^2}-u_2^1u_{22}^2\frac{\partial\xi_2}{\partial u^2}\,. \qquad (6.146)$$

The first invariance condition is

$$X^{(1)}(u_1^1+u_2^2)=0 \quad \rightarrow \quad \eta_1^{(1)1}+\eta_2^{(1)2}=0\ . \qquad (6.147)$$

Using $u_2^2=-u_1^1$ and substituting from (6.141) and (143), separating with respect to higher order variables

$$\frac{\partial\eta^1}{\partial x_1}+\frac{\partial\eta^2}{\partial x_2}=0\,, \qquad (6.148)$$

$$\frac{\partial\eta^1}{\partial u^1}-\frac{\partial\xi_1}{\partial x_1}-\frac{\partial\eta^2}{\partial u^2}+\frac{\partial\xi_2}{\partial x_2}=0\,, \qquad (6.149)$$

$$\frac{\partial\eta^1}{\partial u^2}-\frac{\partial\xi_1}{\partial x_2}=0\,, \qquad (6.150)$$

$$\frac{\partial\eta^2}{\partial u^1}-\frac{\partial\xi_2}{\partial x_1}=0\,, \qquad (6.151)$$

$$\frac{\partial\xi_1}{\partial u^1}+\frac{\partial\xi_2}{\partial u^2}=0\,. \qquad (6.152)$$

The second invariance condition

$$X^{(2)}(u^1u_1^1+u^2u_2^1)=X^{(2)}(UU'+u_{22}^1)\,, \qquad (6.153)$$

yields

$$u_1^1\eta^1+u_2^1\eta^2+u^1\eta_1^{(1)1}+u^2\eta_2^{(1)1}=\xi_1\frac{\partial}{\partial x_1}(UU')+\eta_{22}^{(2)1}\,. \qquad (6.154)$$

The invariance condition can now be calculated and $u_{22}^1=u^1u_1^1+u^2u_2^1-UU'$ and $u_2^2=-u_1^1$ has to be submitted wherever u_{22}^1 and u_2^2 appears. The block of equation can be separated with respect to higher order variables. The coefficient of $u_2^1u_{22}^2$ requires $\frac{\partial\xi_2}{\partial u^2}=0$, the coefficient of $u_1^1u_{22}^2$ requires $\frac{\partial\xi_1}{\partial u^2}=0$, the coefficient of $u_1^1 u_{21}^1$ requires $\frac{\partial \xi_1}{\partial u^2} = 0$, the coefficient of $u_2^1 u_{21}^1$ requires $\frac{\partial \xi_1}{\partial u^1} = 0$, the coefficient of u_{22}^2 requires $\frac{\partial \eta^1}{\partial u^2} = 0$ and the coefficient of u_{21}^1 requires $\frac{\partial \xi_1}{\partial x_2} = 0$. Hence,

$$\xi_1 = \xi_1(x_1), \;\; \xi_2 = \xi_2(x_1, x_2, u^1)\,, \;\; \eta^1 = \eta^1(x_1, x_2, u^1)\,, \tag{6.155}$$

simplifies substantially the second block of equations. The separated equations corresponding to the second block as well as those corresponding to the simplified first block, i.e., Equations (6.148)-(6.152) are listed below

$$\frac{\partial \eta^1}{\partial x_1} + \frac{\partial \eta^2}{\partial x_2} = 0\,, \tag{6.156}$$

$$\frac{\partial \eta^1}{\partial u^1} - \xi_1'(x_1) - \frac{\partial \eta^2}{\partial u^2} + \frac{\partial \xi_2}{\partial x_2} = 0\,, \tag{6.157}$$

$$\frac{\partial \eta^2}{\partial u^1} - \frac{\partial \xi_2}{\partial x_1} = 0\,, \tag{6.158}$$

$$u^1 \frac{\partial \eta^1}{\partial x_1} + u^2 \frac{\partial \eta^1}{\partial x_2} = \xi_1 \frac{\partial}{\partial x_1}(UU') + \frac{\partial^2 \eta^1}{\partial x_2^2} - UU'\left(\frac{\partial \eta^1}{\partial u^1} - 2\frac{\partial \xi_2}{\partial x_2}\right), \tag{6.159}$$

$$\eta^1 = u^1\left(\xi_1'(x_1) - 2\frac{\partial \xi_2}{\partial x_1}\right), \tag{6.160}$$

$$\eta^2 - u^1 \frac{\partial \xi_2}{\partial x_1} = 2\frac{\partial^2 \eta^1}{\partial x_2 \partial u^1} - \frac{\partial^2 \xi_2}{\partial x_2^2} - u^2 \frac{\partial \xi_2}{\partial x_2} + 3UU' \frac{\partial \xi_2}{\partial u^1}\,, \tag{6.161}$$

$$0 = \frac{\partial^2 \xi_2}{\partial x_2 \partial u^1} + 2u^1 \frac{\partial \xi_2}{\partial u^1}\,, \tag{6.162}$$

$$0 = -2\frac{\partial^2 \xi_2}{\partial x_2 \partial u^1} + \frac{\partial^2 \eta^1}{\partial (u^1)^2} - 2u^2 \frac{\partial \xi_2}{\partial u^1}\,, \tag{6.163}$$

$$0 = \frac{\partial^2 \xi_2}{\partial (u^1)^2}\,. \tag{6.164}$$

Solving the above over-determined system of equations and returning back to the original variables, the infinitesimals are

$$\xi_1 = ax + b\,, \tag{6.165}$$

$$\xi_2 = (a + c)y + d(x)\,, \tag{6.166}$$

$$\eta^1 = -(a+2c)u \ , \tag{6.167}$$

$$\eta^2 = -(a+c)v + d'(x)u \ , \tag{6.168}$$

with the function $U(x)$ satisfying

$$(ax+b)\frac{d}{dx}(UU') + (3a+4c)UU' = 0 \ . \tag{6.169}$$

If $U(x)$ satisfies identically the above equation, i.e. none of the parameters a, b and c are nonzero, then one can say that there are 3 finite parameter Lie group of transformations and 1 infinite parameter Lie group of transformations. The base generators are

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_2 = \frac{\partial}{\partial x},$$

$$X_3 = y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad X_\infty = d(x)\frac{\partial}{\partial y} + d'(x)u\frac{\partial}{\partial v} \ . \tag{6.170}$$

The scaling symmetries X_1 and X_3 are compatible with those found in Problem 2.8 using the direct application of the scaling transformation. Indeed, this problem is a kind of group classification problem which will be retreated in Chapter 8.

6.7. EXERCISES

E.6.1. Determine the second order extended infinitesimals and generators for the three parameter spiral group with $u = u(x,y)$

$$x^* = x + \epsilon a \ ,$$
$$y^* = y + \epsilon b y \ ,$$
$$u^* = u + \epsilon c u \ .$$

E.6.2. Determine the second order extended infinitesimals and generators for the three parameter group with $u = u(x,y)$

$$x^* = x + \epsilon a x^2 \ ,$$
$$y^* = y + \epsilon b x y \ ,$$
$$u^* = u + \epsilon c u \ .$$

E.6.3. Consider the first order partial differential equation

$$u_t = u_x^2\ .$$

Calculate the infinitesimals and the base generators admitting the equation.

E.6.4. Consider the dimensionless heat conduction equation

$$u_t = u_{xx}\ .$$

Calculate the infinitesimals and the base generators admitting the equation.

E.6.5. Consider the dimensionless wave equation

$$u_{tt} = u_{xx}$$

Calculate the infinitesimals and the base generators admitting the equation.

E.6.6. Consider the Harry-Dym equation (Basarab-Horwath, 2013)

$$u_t = u^3 u_{xxx}\ .$$

Calculate the infinitesimals and the base generators admitting the equation.

E.6.7. Consider the coupled equation of filtration (Pakdemirli, 2002)

$$c_x + \lambda c = 0, \quad \sigma_t + v c_x = 0$$

where x is the depth, t is the time, $c(x,t)$ is the concentration, λ is the filter constant, $\sigma(x,t)$ is the specific deposit and v is the constant approach velocity. Calculate the infinitesimals and the base generators admitting the equation.