CHAPTER 5

ORDINARY DIFFERENTIAL EQUATIONS: SYMMETRY SOLUTIONS

In this chapter, constructing solutions using the symmetry generators will be discussed separately for first, second and higher order ordinary differential equations. Symmetries can be employed in a number of ways: Determining integrating factors (first order equations), group invariant solutions (discussed in Chapter 2 and will be discussed more systematically here), reduction of order via canonical coordinates, reduction of order via group invariants and finding another solution from a known one via the group transformations.

A one parameter Lie Group of transformation may be employed to reduce the order of equation by one. For second order equations, if there exists two or more symmetries, it is always possible to reduce the order of equation by two thereby obtaining full reduction. This is because any two dimensional Lie algebra is solvable and the requirement for successive reductions is the solvability of the Lie algebras. For higher order equations, if there are *r* symmetries and the order is *k*, reduction to order *k*-*r* is only possible if a solvable Lie algebra can be constructed.

5.1. FIRST ORDER DIFFERENTIAL EQUATIONS

For first order differential equations, the relationship of symmetries with the integrating factors will be outlined first. Group invariant solutions, reduction of order via canonical coordinates will be discussed next. Finally, mapping a solution from a known one will be treated.

Integrating Factors

A common method to integrate nonlinear first order equations of the form

$$M(x,y)dx + N(x,y)dy = 0 \,, \tag{5.1}$$

is the method of integrating factors (O'Neil, 1991). If the equation is not in the exact differential form, then the equation is multiplied by an integrating factor $\mu(x,y)$ to make it exact

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0\ , \qquad (5.2)$$

where now

$$\frac{\partial\omega}{\partial x} = \mu M,\ \frac{\partial\omega}{\partial y} = \mu N\ \ , \qquad (5.3)$$

so that (5.2) is an exact differential

$$\frac{\partial\omega}{\partial x}dx + \frac{\partial\omega}{\partial y}dy = 0 \quad \rightarrow \quad d\omega = 0 \quad , \qquad (5.4)$$

with a solution

$$\omega(x,y) = c\ , \qquad (5.5)$$

where c is a constant of integration. To determine the integrating factor, one uses the interchangeability of the derivative orders, hence $\frac{\partial^2\omega}{\partial x\partial y} = \frac{\partial^2\omega}{\partial y\partial x}$ or from (5.3)

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)\ . \qquad (5.6)$$

To outline the relationship with integrating factors and symmetries, write (5.4) as

$$\frac{\partial\omega}{\partial x} + \frac{\partial\omega}{\partial y}y' = 0\ . \qquad (5.7)$$

But from (5.1) $y' = -M/N$ and upon substituting and multiplying by N yields

$$N\frac{\partial\omega}{\partial x} - M\frac{\partial\omega}{\partial y} = 0\ . \qquad (5.8)$$

One now requires the invariance condition $X\omega = 1$ for family of curves (See Eq. 3.62)

$$\xi\frac{\partial\omega}{\partial x} + \eta\frac{\partial\omega}{\partial y} = 1\ . \qquad (5.9)$$

Solving (5.8) and (5.9) together

$$\frac{\partial\omega}{\partial x} = \frac{\begin{vmatrix} 0 & -M \\ 1 & \eta \end{vmatrix}}{\begin{vmatrix} N & -M \\ \xi & \eta \end{vmatrix}} = \frac{M}{\xi M + \eta N}\ , \qquad (5.10)$$

$$\frac{\partial \omega}{\partial y} = \frac{\begin{vmatrix} N & 0 \\ \xi & 1 \end{vmatrix}}{\begin{vmatrix} N & -M \\ \xi & \eta \end{vmatrix}} = \frac{N}{\xi M + \eta N} \quad . \tag{5.11}$$

But since $d\omega = \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy = 0$ is an exact differential, substituting the equivalents of the derivatives

$$\frac{M}{\xi M + \eta N}\, dx + \frac{N}{\xi M + \eta N} dy = 0\ , \tag{5.12}$$

clearly shows that

$$\mu = \frac{1}{\xi M + \eta N}\quad , \tag{5.13}$$

is the integrating factor for the equation admitting a symmetry generator $X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$. Note that if μ is an integrating factor, then $k\mu$ is also an integrating factor for some constant k. This fact can be verified by multiplying (5.12) with a constant k.

Problem 5.1. Consider the nonlinear first order differential equation solved in Problem (4.4).

$$y' + y^2 = 0\ . \tag{5.14}$$

Determine the integrating factor for the equation and solve the equation.

Solution

The equation can be written in the form

$$y^2 dx + dy = 0\ . \tag{5.15}$$

Hence $M = y^2$ and $N = 1$. One of the symmetries of the equation was already given in Problem 4.4 as $\xi = 1$ and $\eta = y^2$. Substitute all into (5.13)

$$\mu = \frac{1}{y^2}\, , \tag{5.16}$$

without considering the factor ½. If one multiplies (5.15) with the integrating factor

$$dx + \frac{dy}{y^2} = 0\ , \tag{5.17}$$

the equation can be directly integrable with a result

$$x - \frac{1}{y} = c \quad \rightarrow \quad y = \frac{1}{x-c} \,. \tag{5.18}$$

Problem 5.2. Consider the first order differential equation

$$2ydx + 3xdy = 0 \,. \tag{5.19}$$

Determine the integrating factor for the equation and solve the equation.

Solution

The equation can be written in the form

$$y' + \frac{2}{3}\frac{y}{x} = 0 \,. \tag{5.20}$$

It is easy to show that the equation admits uniform scaling. Hence $\xi = x$ and $\eta = y$. From (5.19), $M = 2y$ and $N = 3x$. Substitute all into (5.13)

$$\mu = \frac{1}{xy} \,, \tag{5.21}$$

without considering the factor 1/5 for simplicity. If one multiplies (5.19) with the integrating factor

$$2\frac{dx}{x} + 3\frac{dy}{y} = 0 \,, \tag{5.22}$$

the equation can be directly integrable with a solution

$$x^2 y^3 = c \,. \tag{5.23}$$

Group Invariant Solutions

Such solutions were the special type of solutions treated in Chapter 2 when solving ODEs. If the symmetry generator is $X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$, then the group invariant solutions are the solutions which satisfy both the original equation and the characteristic equation

$$\frac{dx}{\xi} = \frac{dy}{\eta} \quad \text{or} \quad y' = \frac{\eta}{\xi} \,. \tag{5.24}$$

For the given symmetries, the solutions in Problem 5.1 and 5.2 are not group invariant solutions since although they satisfy the original equations, they do not satisfy the associated determining equations given in (5.24).

Group invariant solutions can also be defined as the solution curves which are invariant under the symmetry generators or the symmetry groups admitted by the equation (Definition 3.1 and Theorem 3.1). Solutions different from the group invariant solutions can be considered those solutions which leaves the family of curves invariant according to Definition 3.2 and Theorem 3.2.

Problem 5.3. Consider the nonlinear first order differential equation

$$y' - y^2 = 0\,. \tag{5.25}$$

Determine a group invariant solution using the scaling symmetry.

Solution

Using the methods of Chapter 2, it is easy to show that for the scaling group transformations $x^* = e^{\epsilon a}x$ and $y^* = e^{\epsilon b}y$, the parameters should satisfy $b = -a$. Hence, the non-uniform one parameter scaling transformation is $x^* = e^{\epsilon}x$ and $y^* = e^{-\epsilon}y$ by choosing $a = 1$ without loss of generality. Alternatively, the general determining equation (4.53) given in the previous chapter can be used with an assumption of $\xi = x$ to determine $\eta = \eta(y)$.The corresponding generator is $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. From (5.24)

$$\frac{dx}{x} = \frac{dy}{-y}\,, \tag{5.26}$$

the equation can be directly integrable with a result

$$y = \frac{c}{x}\,. \tag{5.27}$$

This solution should also satisfy the original equation. Substituting into (5.25), $c = 0$ or $c = -1$. Discarding the trivial solution for $c = 0$, the non-trivial group invariant solution associated with the non-uniform scaling symmetry is

$$y = -\frac{1}{x}\,. \tag{5.28}$$

Reduction of Order by Canonical Coordinates

Any infinitesimal Lie group of transformations

$$x^* = x + \epsilon\xi(x,y), \;\; y^* = y + \epsilon\eta(x,y) \tag{5.29}$$

can be transformed into a translational symmetry in the dependent variable via some coordinate transformations

$$r^* = r, \qquad s^* = s + \epsilon \tag{5.30}$$

where $r = r(x,y)$ and $s = s(x,y)$. Then the original symmetry generator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\,, \tag{5.31}$$

transforms into a much simpler form

$$X = \frac{\partial}{\partial s}\,, \tag{5.32}$$

where r and s are called the canonical coordinates and are determined by the equations

$$Xr = 0\,, \;\; Xs = 1\,. \tag{5.33}$$

From above, one forces the first infinitesimal to disappear and the second infinitesimal to be 1 in view of (5.30) and (5.32). Using canonical coordinates, the order of the equation can be reduced.

Problem 5.4. Re-consider the nonlinear first order differential equation of Problem 5.3

$$y' - y^2 = 0\,. \tag{5.34}$$

Using canonical coordinates corresponding to the scaling symmetry, reduce the order and find a solution.

Solution

The infinitesimal generator was found in the previous problem as $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ for scaling symmetry. From the first equation of (5.33)

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dr}{0}\,, \tag{5.35}$$

which yields

$$r = xy \ . \tag{5.36}$$

From the second equation of (5.33)

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{ds}{1} \ , \tag{5.37}$$

and taking $ds = \frac{dx}{x}$

$$s = \ln x \ . \tag{5.38}$$

Calculate $\frac{ds}{dr}$,

$$\frac{ds}{dr} = \frac{s_x + s_y y'}{r_x + r_y y'} \ . \tag{5.39}$$

Since $s_x = 1/x$, $s_y = 0$, $r_x = y$, $r_y = x$ and $y' = y^2$

$$\frac{ds}{dr} = \frac{1}{xy + x^2y^2} = \frac{1}{r(1+r)} = \frac{1}{r} - \frac{1}{(1+r)} \ . \tag{5.40}$$

The right hand side transformed into a pure function of the independent variable *r* which can be directly integrated

$$s = \ln\left(c\,\frac{r}{r+1}\right) . \tag{5.41}$$

Substituting the equivalents of *r* and *s* in (5.36) and (5.38) and solving for *y*

$$y = \frac{1}{c-x} \ . \tag{5.42}$$

For this specific example, the equation is separable and directly integrable without coordinate transformations. However, outlining the essential steps of canonical transformations is important for more involved problems which cannot be directly integrated.

Problem 5.5. Using canonical coordinates, solve the linear non-homogenous first order differential equation

$$y' + a(x)y = h(x) \ , \tag{5.43}$$

by first verifying that an infinite parameter symmetry of the equation is $X = u(x)\frac{\partial}{\partial y}$ where $u(x)$ is the solution of the homogenous problem.

Solution

Apply the extended generator to the equation

$$\left(u(x)\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y_1}\right)(y_1 + a(x)y - h(x)) = 0 \quad , \tag{5.44}$$

with the invariance condition being

$$\eta^{(1)} + u(x)a(x) = 0 \quad . \tag{5.45}$$

From (4.23)

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y_1 - \xi_y y_1^2 \,, \tag{5.46}$$

with $\xi = 0$ and $\eta = u$

$$\eta^{(1)} = u'(x) \,. \tag{5.47}$$

Hence, the invariance condition (5.45) requires

$$u'(x) + u(x)a(x) = 0 \quad , \tag{5.48}$$

which is exactly the homogenous equation with a solution

$$u(x) = e^{-\int a(x)dx} \,. \tag{5.49}$$

From the first equation of (5.33), $Xr = 0$,

$$u\frac{\partial r}{\partial y} = 0 \,, \tag{5.50}$$

and a simple particular solution may be

$$r = x \,. \tag{5.51}$$

From the second equation of (5.33), $Xs = 1$

$$u\frac{\partial s}{\partial y} = 1 \,, \tag{5.52}$$

with

$$s = \frac{y}{u} \,. \tag{5.53}$$

Calculate $\frac{ds}{dr}$,

$$\frac{ds}{dr} = \frac{s_x + s_y y'}{r_x + r_y y'} = \frac{-\frac{y}{u^2}u' + \frac{1}{u}y'}{1} = -s\,\frac{u'}{u} + \frac{y'}{u}\,, \tag{5.54}$$

with $u = u(r)$. Solving y' from above and substituting into the original differential equation

$$u(r)\,\frac{ds}{dr} + s\left(\frac{du}{dr} + a(r)u(r)\right) = h(r)\;. \tag{5.55}$$

But the terms in the parenthesis cancel since u is the homogenous solution leaving

$$u(r)\,\frac{ds}{dr} = h(r)\;. \tag{5.56}$$

Dividing by $u(r)$ and integrating

$$s = \int \frac{h(r)}{u(r)}\,dr + c\;. \tag{5.57}$$

Returning back to the original variables and using (5.49)

$$y(x) = e^{-\int a(x)dx}\left(\int e^{\int a(x)dx} h(x)dx + c\right)\,, \tag{5.58}$$

which is the general solution of a linear non-homogenous first order equation. As can be seen, many of the ad-hoc techniques indeed inherit symmetry solutions of equations making the symmetry analysis a unified and generalized approach.

Mapping a Solution into Another Solution

If one knows a solution of the problem, then from that solution, another solution can be constructed via the symmetry transformations. The restriction for mapping solutions to other solutions is that the first solution should not be a group invariant solution. Otherwise, the solution will map exactly onto it without producing a different solution.

Problem 5.6. Re-consider the nonlinear first order differential equation of Problem 5.4 again

$$y' - y^2 = 0\;. \tag{5.59}$$

Using the non-uniform scaling transformation admitted by the equation, find another solution starting from the solution $y = \frac{1}{c-x}$.

Solution

Since the solution is not a group invariant solution

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(y - \frac{1}{c-x}\right) = -\frac{x}{(c-x)^2} - y = -\frac{x}{(c-x)^2} - \frac{1}{c-x} \neq 0\ , \tag{5.60}$$

it can be mapped into another solution. Substitute the non-uniform scalings $x^* = e^{\epsilon}x$ and $y^* = e^{-\epsilon}y$ admitted by the original differential equation into the solution $y = \frac{1}{c-x}$

$$e^{\epsilon}y^* = \frac{1}{c-e^{-\epsilon}x^*}\ , \tag{5.61}$$

and reorganizing

$$y^* = \frac{1}{c^*-x^*}\ , \tag{5.62}$$

where $c^* = ce^{\epsilon}$. On the contrary, one can show that the group invariant solution $y = -\frac{1}{x}$ maps onto itself under the transformation, i.e. $y^* = -\frac{1}{x^*}$ without producing a new solution.

5.2. SECOND ORDER DIFFERENTIAL EQUATIONS

For the second order differential equations, symmetries can be applied in a number of different ways. Integrating factor method will not be discussed although integrating factors can be employed for arbitrary orders of the differential equations. A systematic way of determining integrating factors for arbitrary orders of ODEs were outlined by Pakdemirli (2023a) using the principles of variational calculus. The group invariant methods will be discussed first. Then reduction of order via canonical coordinates and differential invariants will be discussed. As a general rule, a single symmetry reduces the order of equation by one. Special to second order equations, if the equation possesses two or more symmetries, then complete reduction is possible. This is because Lie algebra of order two is always solvable which is not always the case for higher order equations (order three or more). Mapping a solution from a known one is another option where symmetries can be employed.

Group Invariant Solutions

As mentioned in Section 5.1, group invariant solutions are the solutions which satisfy both the original differential equation and the determining equation

$$\frac{dx}{\xi} = \frac{dy}{\eta} \quad \text{or} \quad y' = \frac{\eta}{\xi} \ , \tag{5.63}$$

where $X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ is the symmetry generator of the equation itself. It may happen that both equations cannot be satisfied at all or possess trivial solutions as common solutions. In such cases, one cannot speak of group-invariant solutions. In an alternative method presented by Bluman and Kumei (1989), the need of explicitly calculating (5.63) can be waived.

Problem 5.7. Consider the differential equation describing the path of the object subject to minimum drag force

$$y'' - \alpha(1 + y'^2) = 0 \ . \tag{5.64}$$

Among the eight finite parameter symmetries, infinitesimals corresponding to two of the symmetries of the equation were (Pakdemirli and Aksoy, 2010)

$$\xi = (a\cos\alpha x + b\sin\alpha x)e^{-\alpha y}, \ \eta = (a\sin\alpha x - b\cos\alpha x)e^{-\alpha y} \tag{5.65}$$

Construct a group invariant solution using the above infinitesimals.

Solution

From (5.63)

$$\frac{dx}{(a\cos\alpha x + b\sin\alpha x)e^{-\alpha y}} = \frac{dy}{(a\sin\alpha x - b\cos\alpha x)e^{-\alpha y}} \ , \tag{5.66}$$

or

$$y' = \frac{a\sin\alpha x - b\cos\alpha x}{a\cos\alpha x + b\sin\alpha x} , \tag{5.67}$$

which can be directly integrated

$$y = -\frac{1}{\alpha}\ln(a\cos\alpha x + b\sin\alpha x) + c \ . \tag{5.68}$$

Substituting this solution into (5.64), no further restrictions appear on the parameters *a*, *b* and *c*. Hence, (5.68) is a group-invariant solution.

Problem 5.8. Consider the nonlinear second order differential equation

$$y'' + y'^2 = 0 \ . \tag{5.69}$$

All symmetries were already calculated in Problem 4.6. Calculate a group invariant solution corresponding to the symmetries $X_2 = x\frac{\partial}{\partial x}$, $X_6 = xe^{-y}\frac{\partial}{\partial y}$.

Solution

The symmetry generator corresponding to the two symmetries is $X = ax\frac{\partial}{\partial x} + bxe^{-y}\frac{\partial}{\partial y}$ for some constant parameters a and b. From (5.63)

$$\frac{dx}{ax} = \frac{dy}{bxe^{-y}} \ , \tag{5.70}$$

or

$$e^{y}dy = k_1 dx \ , \tag{5.71}$$

where $k_1 = b/a$. Integrating and solving for y

$$y = \ln(k_1 x + k_2) \ . \tag{5.72}$$

Substituting this solution into (5.69), no further restrictions appear on the constants k_1 and k_2. Hence, (5.72) is a group-invariant solution.

As mentioned earlier, mapping a group invariant solution to another solution is impossible because the associated symmetries will produce exactly the same solution.

Reduction of Order by Canonical Coordinates

In the previous section, the necessary equations to determine the canonical coordinates were already given, i.e. Equation (5.33). If one symmetry of the equation is known, the second order equation can be transferred into a first order equation. If at least two symmetries are known, then the equation may be reduced to an algebraic equation which will be the solution of the problem.

Problem 5.9. Using canonical coordinates, reduce the order of the linear homogenous variable coefficient second order ordinary differential equation

$$y'' + p(x)y' + q(x)y = 0 \ , \tag{5.73}$$

by using the symmetry $x^* = x$, $y^* = e^{\epsilon} y$.

Solution

The infinitesimals are $\xi = 0$ and $\eta = y$ and the generator is $X = y\frac{\partial}{\partial y}$. From the first equation of (5.33) $Xr = 0$

$$y\frac{\partial r}{\partial y} = 0 \,, \tag{5.74}$$

and a simple particular solution may be

$$r = x \,. \tag{5.75}$$

From the second equation of (5.33), $Xs = 1$

$$y\frac{\partial s}{\partial y} = 1 \,, \tag{5.76}$$

with

$$s = \ln y \,. \tag{5.77}$$

Calculate $\frac{ds}{dr}$,

$$\frac{ds}{dr} = \frac{s_x + s_y y'}{r_x + r_y y'} = \frac{\frac{1}{y} y'}{1} = \frac{y'}{y} \,, \tag{5.78}$$

with $s = s(r)$. Solving y'

$$y' = y\frac{ds}{dr} \,, \tag{5.79}$$

and differentiating once more

$$y'' = y'\frac{ds}{dr} + y\frac{d^2 s}{dr^2} = y\left(\frac{ds}{dr}\right)^2 + y\frac{d^2 s}{dr^2} \,. \tag{5.80}$$

Substituting (5.80) and (5.79) into the original differential equation and dividing by y

$$\frac{d^2 s}{dr^2} + \left(\frac{ds}{dr}\right)^2 + p(r)\frac{ds}{dr} + q(r) = 0 \,. \tag{5.81}$$

Defining $u = \frac{ds}{dr}$, the above equation reduces to the well-known first order Riccati equation

$$\frac{du}{dr} + u^2 + p(r)u + q(r) = 0 \ . \tag{5.82}$$

Transformations similar to the canonical coordinates presented here are given in textbooks on differential equations (O'Neil, 1991). The transformations are found by ad hoc methods whereas here, a systematic derivation of the transformation was given by symmetry approaches. For solutions of Riccati equations, see O'Neil (1991) and Ndiaye (2022) for example. When the differential operator can be decomposed into core and shell components, the general solution for (5.73) can be written immediately by the newly proposed core-shell approach (Pakdemirli, 2024a).

Problem 5.10. Using canonical coordinates, solve the nonlinear second order ordinary differential equation

$$y'' - 2yy' = 0 \ , \tag{5.83}$$

by using the symmetry generator $X = \frac{\partial}{\partial x}$ found in Problem 4.7.

Solution

From the first equation of (5.33), $Xr = 0$,

$$\frac{\partial r}{\partial x} = 0 \ , \tag{5.84}$$

and a simple particular solution may be

$$r = y \ . \tag{5.85}$$

From the second equation of (5.33), $Xs = 1$,

$$\frac{\partial s}{\partial x} = 1 \ , \tag{5.86}$$

with

$$s = x \ . \tag{5.87}$$

Calculate $\frac{ds}{dr}$,

$$\frac{ds}{dr} = \frac{s_x + s_y y'}{r_x + r_y y'} = \frac{1}{y'} \,, \tag{5.88}$$

with $s = s(r)$. The second derivative is calculated next

$$\frac{d^2 s}{dr^2} = \frac{\left(\frac{ds}{dr}\right)_x + \left(\frac{ds}{dr}\right)_y y' + \left(\frac{ds}{dr}\right)_{y'} y''}{r_x + r_y y'} = \frac{-\frac{y''}{y'^2}}{y'} = -\frac{y''}{y'^3} \,, \tag{5.89}$$

Hence

$$y = r, \quad y' = \frac{1}{\frac{ds}{dr}} \,, \quad y'' = -\frac{\frac{d^2 s}{dr^2}}{\left(\frac{ds}{dr}\right)^3} \,. \tag{5.90}$$

Substituting to the original equation and rearranging

$$\frac{d^2 s}{dr^2} + 2r \left(\frac{ds}{dr}\right)^2 = 0 \,. \tag{5.91}$$

Defining $u = \frac{ds}{dr}$, the above equation reduces to the first order equation

$$\frac{du}{dr} + 2ru^2 = 0 \,, \tag{5.92}$$

which is separable

$$\frac{du}{u^2} = -2r dr \,, \tag{5.93}$$

with a solution

$$u = \frac{ds}{dr} = \frac{1}{c_1^2 + r^2} \,. \tag{5.94}$$

Solving for s,

$$s = \frac{1}{c_1} \text{Arctan}\left(\frac{r}{c_1}\right) - c_2 \,. \tag{5.95}$$

Since $s = x$ and $r = y$, substituting the original variables and solving for y

$$y = c_1 \tan[c_1(x + c_2)] \,. \tag{5.96}$$

Reduction of Order by Differential Invariants

A similar approach to canonical coordinates is to reduce the order by employing differential invariants. For second order equations, instead of employing (5.33), the invariants are calculated from the system of determining equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy_1}{\eta^{(1)}} \,. \tag{5.97}$$

Problem 5.11. Reconsider Problem 5.10. Using differential invariants, solve the nonlinear second order ordinary differential equation

$$y'' - 2yy' = 0 \,, \tag{5.98}$$

by using the symmetry generator $X = \frac{\partial}{\partial x}$ found in Problem 4.7.

Solution

The infinitesimals are $\xi = 1$, $\eta = 0$ and $\eta^{(1)} = 0$. From (5.97),

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy_1}{0} \,. \tag{5.99}$$

Taking the first two equations, the first invariant is

$$u = y \,. \tag{5.100}$$

From the last two equations, the second invariant is

$$v = y_1 \,. \tag{5.101}$$

Calculate $\frac{dv}{du}$,

$$\frac{dv}{du} = \frac{v_x + v_y y_1 + v_{y_1} y_2}{u_x + u_y y_1} = \frac{y_2}{y_1} \,, \tag{5.102}$$

Hence $y_2 = v\frac{dv}{du}$, $y_1 = v$ and $y = u$. Substituting into the original equation and dividing by v

$$\frac{dv}{du} - 2u = 0 \,, \tag{5.103}$$

which is separable

$$dv = 2udu \,, \tag{5.104}$$

with a solution

$$v = c_1^2 + u^2 \; . \tag{5.105}$$

Since $v = \frac{dy}{dx}$ and $u = y$,

$$\frac{dy}{dx} = c_1^2 + y^2 \; , \tag{5.106}$$

which is separable

$$\frac{dy}{c_1^2+y^2} = dx \;\; , \tag{5.107}$$

with a solution

$$\frac{1}{c_1}\mathrm{Arctan}\left(\frac{y}{c_1}\right) = x + c_2 \;\; . \tag{5.108}$$

Solving for y, the same result is obtained

$$y = c_1 \tan[c_1(x + c_2)] \; . \tag{5.109}$$

5.3. HIGHER ORDER DIFFERENTIAL EQUATIONS

For the equations with orders equal to three or greater, complete reduction is subject to the existence of a solvable Lie algebra. However, even one or two reductions may simplify the equation enabling it to be solved either analytically or numerically. For the direct method of determining group invariant solutions, no order reductions are necessary and the solutions are found at once. However, such solutions are restricted solutions and may not contain enough parameters to satisfy the conditions imposed on the differential equations. Reduction can be done via canonical coordinates or via differential invariants. One or two order reductions are guaranteed but further reductions are subject to the existence of a solvable Lie algebra. Mapping a solution from a known one is another option where symmetries can be employed.

Group Invariant Solutions

As mentioned in the previous sections, group invariant solutions are the solutions which satisfy both the original differential equation and the determining equation

$$\frac{dx}{\xi} = \frac{dy}{\eta} \quad \text{or} \quad y' = \frac{\eta}{\xi} \;, \tag{5.110}$$

where $X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ is the symmetry generator of the equation itself. Depending on the symmetry, such solutions may exist, may be trivial or simple solutions or may not exist at all. Two sample problems are treated.

Problem 5.12. Consider the Blasius equation appearing in boundary layer equations

$$y''' + \frac{1}{2} y y'' = 0 \ . \tag{5.111}$$

The equation admits two parameter Lie group of transformations with infinitesimals (Bluman and Kumei, 1989),

$$\xi = ax + b \ , \quad \eta = -ay \ . \tag{5.112}$$

Find a group invariant solution corresponding to the above symmetries.

Solution

From (5.110)

$$\frac{dx}{ax+b} = \frac{dy}{-ay} \quad . \tag{5.113}$$

Solving the above equation

$$y = \frac{c_1}{ax+b} \ , \tag{5.114}$$

and substituting into the original equation yields

$$c_1 a^2(-6a + c_1) = 0 \ . \tag{5.115}$$

Assume $c_1 a^2 \neq 0$ for nontrivial solutions. Hence $c_1 = 6a$ and the final solution satisfying both the characteristic equation and the original equation is

$$y = \frac{6}{x+c} \ , \tag{5.116}$$

where $c = b/a$.

Problem 5.13. Consider the constant normal acceleration curve equation

$$(1 + y'^2) y''' - 3y' y''^2 = 0 \ . \tag{5.117}$$

The equation admits 6 parameter Lie group of transformations with base generators (Pakdemirli, 2023a)

$$X_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$X_5 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y},\ \ X_6 = -2xy\frac{\partial}{\partial x} + (x^2 - y^2)\frac{\partial}{\partial y}\ . \tag{5.118}$$

Find a group invariant solution corresponding to the X_2, X_3 and X_4 base generators.

Solution

The infinitesimals corresponding to the base generators X_2, X_3 and X_4 are

$$\xi = ax + b\ , \quad \eta = ay + c\ . \tag{5.119}$$

From (5.110)

$$\frac{dx}{ax+b} = \frac{dy}{ay+c}\ . \tag{5.120}$$

Solving the above equation

$$y = c_1 x + c_2\ , \tag{5.121}$$

for some constants $c_1$and c_2 related to the parameters. Since the solution satisfies the original equation without further restrictions, the solution is the group-invariant solution.

Reduction of Order by Differential Invariants

Differential invariants method is employed for reducing the order of equations. For a third order differential equation, the invariants are calculated form the characteristic equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy_1}{\eta^{(1)}} = \frac{dy_2}{\eta^{(2)}}\ . \tag{5.122}$$

Two sample problems will be solved. In each problem, by using two symmetry base generators, the equations are reduced to first order equations.

Problem 5.14. Consider again the Blasius equation

$$y''' + \frac{1}{2} y y'' = 0 \quad . \tag{5.123}$$

The Lie algebra for the equation is

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \quad . \tag{5.124}$$

Using both generators, reduce the order of the equation by 2 using differential invariants method.

Solution

By successive application of the generators, the equation can be cast into a first order ODE. Since $[X_1, X_2] = X_1$, the solvable Lie algebra is $\mathcal{L}^1 = (X_1)$ and $\mathcal{L}^2 = (X_1, X_2)$ with $\mathcal{L}^1 \subset \mathcal{L}^2$. One should start with X_1 first and then apply X_2. Since $\eta^{(1)} = 0$ for the first generator, the first extension of it is $X_1^{(1)} = \frac{\partial}{\partial x}$. The characteristic equation corresponding to $X_1$is therefore

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy_1}{0} \quad . \tag{5.125}$$

The invariants of the system are

$$u = y, \qquad v = y_1 \; . \tag{5.126}$$

The derivative is another invariant to be calculated

$$\dot{v} = \frac{dv}{du} = \frac{v_x + v_y y_1 + v_{y_1} y_2}{u_x + u_y y_1} = \frac{y_2}{y_1} \, . \tag{5.127}$$

The second generator has to be expressed in terms of the differential invariants. Since $\eta^{(1)} = -2y_1$ and $\eta^{(2)} = -3y_2$ (See Problem 4.2), the extension of the second generator to second order is

$$X_2^{(2)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2y_1\frac{\partial}{\partial y_1} - 3y_2\frac{\partial}{\partial y_2} \, . \tag{5.128}$$

Since the components of the generator in terms of the transformed variables are

$$X_2(u) = -y = -u, X_2^{(1)}(v) = -2y_1 = -2v, X_2^{(2)}(\dot{v}) = -\frac{y_2}{y_1} = -\dot{v} \tag{5.129}$$

the transformed generator can be written as

$$X_2^{(2)} = u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + \dot{v}\frac{\partial}{\partial \dot{v}}\,. \tag{5.130}$$

For the second generator, a further coordinate transformation is needed. If $U = U(u, v)$ and $V = V(u, v)$, then

$$X_2^{(1)}(U) = 0 \quad \rightarrow \quad \frac{du}{u} = \frac{dv}{2v} = \frac{dU}{0}\,, \tag{5.131}$$

yields

$$U = \frac{v}{u^2}\,, \tag{5.132}$$

or

$$U = \frac{y_1}{y^2}\,, \tag{5.133}$$

in terms of the original variables. Similarly

$$X_2^{(2)}(V) = 0 \quad \rightarrow \quad \frac{du}{u} = \frac{dv}{2v} = \frac{d\dot{v}}{\dot{v}} = \frac{dV}{0}\,, \tag{5.134}$$

yields

$$V = \frac{\dot{v}}{u}\,, \tag{5.135}$$

or

$$V = \frac{y_2}{yy_1}\,, \tag{5.136}$$

in terms of the original variables. One can calculate the first derivative with respect to the last transformed variables

$$\frac{dV}{dU} = \frac{d\left(\frac{y_2}{yy_1}\right)}{d\left(\frac{y_1}{y^2}\right)} = \frac{y_3y_1y^2 - y_2^2y^2 - y_2y_1^2y}{y_1^2y_2y - 2y_1^4}\,. \tag{5.137}$$

From the original equation, since $y_3 = -\frac{1}{2}yy_2$, substituting above and simplifying

$$\frac{dV}{dU} = \frac{\frac{1}{2}y^3y_1y_2 + y_2^2y^2 + y_2y_1^2y}{2y_1^4 - y_1^2y_2y}\,. \tag{5.138}$$

From (5.133), $y_1 = Uy^2$ and from (5.136), $y_2 = Vyy_1 = UVy^3$. Substituting into above and simplifying

$$\frac{dV}{dU} = \frac{V}{U}\frac{\frac{1}{2}+V+U}{2U-V} \, . \tag{5.139}$$

The Blasius equation has been transformed into a first order equation. If this equation is solved, then back substitution will yield the analytical solution. The Blasius equation has no known exact analytical solution, hence it is probable that one may stuck when finding an analytical solution of the transformed equation or finding the integrals in back substitution.

Problem 5.15. Consider again the constant normal acceleration curve equation

$$(1 + y'^2)y''' - 3y'y''^2 = 0 \, . \tag{5.140}$$

The equation admits 6 parameter Lie group of transformations with two of the base vectors being (Pakdemirli, 2023a)

$$X_1 = \frac{\partial}{\partial x}, \; X_2 = \frac{\partial}{\partial y} \, . \tag{5.141}$$

Using the above generators, reduce the order of the equation by 2 using differential invariants method.

Solution

The Lie Algebra corresponding to the two base vectors is Abelian and solvable. Since $[X_1, X_2] = 0$, one may start from either of the base generators. Both cases will be investigated.

i) X_1 first and then X_2

Take X_1 first. Since $\eta^{(1)} = 0$ for the first generator, the first extension of it is $X_1^{(1)} = \frac{\partial}{\partial x}$. The characteristic equation corresponding to X_1 is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy_1}{0} \quad . \tag{5.142}$$

The invariants of the system are

$$u = y, \qquad v = y_1 \, . \tag{5.143}$$

The derivative is another invariant to be calculated

$$\dot{v} = \frac{dv}{du} = \frac{v_x + v_y y_1 + v_{y_1} y_2}{u_x + u_y y_1} = \frac{y_2}{y_1} . \tag{5.144}$$

The second generator has to be expressed in terms of the differential invariants. Since $\eta^{(1)} = 0$ and $\eta^{(2)} = 0$, the extension of the second generator to second order is

$$X_2^{(2)} = \frac{\partial}{\partial y} . \tag{5.145}$$

Since the components of the generator in terms of the transformed variables are

$$X_2(u) = 1, X_2^{(1)}(v) = 0, X_2^{(2)}(\dot{v}) = 0 , \tag{5.146}$$

the transformed generator can be written as

$$X_2^{(2)} = \frac{\partial}{\partial u} . \tag{5.147}$$

For the second generator, a further coordinate transformation is needed. If $U = U(u, v)$ and $V = V(u, v)$, then

$$X_2^{(1)}(U) = 0 \quad \rightarrow \quad \frac{du}{1} = \frac{dv}{0} = \frac{dU}{0} , \tag{5.148}$$

yields

$$U = v = y_1 , \tag{5.149}$$

in terms of the original variables. Similarly

$$X_2^{(2)}(V) = 0 \quad \rightarrow \quad \frac{du}{1} = \frac{dv}{0} = \frac{d\dot{v}}{0} = \frac{dV}{0} , \tag{5.150}$$

yields

$$V = \dot{v} , \tag{5.151}$$

or

$$V = \frac{y_2}{y_1} , \tag{5.152}$$

in terms of the original variables. One can calculate the first derivative with respect to the last transformed variables

$$\frac{dV}{dU}=\frac{d\left(\frac{y_2}{y_1}\right)}{d(y_1)}=\frac{y_3y_1-y_2^2}{y_1^2y_2}\,. \qquad (5.153)$$

From the original equation, since $y_3=\frac{3y_1y_2^2}{1+y_1^2}$, substituting above and simplifying

$$\frac{dV}{dU}=\frac{3y_2}{(1+y_1^2)}-\frac{y_2}{y_1^2}\,. \qquad (5.154)$$

From (5.149), $y_1=U$ and from (5.152), $y_2=Vy_1=UV$. Substituting into above and simplifying

$$\frac{dV}{dU}=\frac{3U}{1+U^2}V-\frac{V}{U}\,. \qquad (5.155)$$

which is separable and can be solved. The solution is

$$V=c_1\frac{(1+U^2)^{3/2}}{U}\,. \qquad (5.156)$$

The above solution in terms of the intermediate variables is then

$$\frac{dv}{du}=c_1\frac{(1+v^2)^{3/2}}{v}\,. \qquad (5.157)$$

which can be separated and integrated. In terms of the original variables

$$-\frac{1}{\sqrt{1+y'^2}}=c_1y+c_2\,. \qquad (5.158)$$

i) X_2 first and then X_1

A similar analysis presented above yields the first order equation

$$\frac{dV}{dU}=\frac{3U}{1+U^2}V\,, \qquad (5.159)$$

with

$$V=\frac{dv}{du}=y_2, \quad U=v=y_1, \;\; u=x\,. \qquad (5.160)$$

The solution of (5.159) is

$$V=c_3(1+U^2)^{3/2}\,. \qquad (5.161)$$

The above solution in terms of the intermediate variables is then

$$\frac{dv}{du} = c_3(1+v^2)^{3/2} , \tag{5.162}$$

which can be separated and integrated. In terms of the original variables

$$\frac{y'}{\sqrt{1+y'^2}} = c_3 x + c_4 . \tag{5.163}$$

Squaring (5.158) and (5.163) and adding, the first derivative can be eliminated yielding

$$(c_3 x + c_4)^2 + (c_1 y + c_2)^2 = 1 , \tag{5.164}$$

which shows that the solution curve is a circle. The equation was derived on the assumption of constant normal acceleration with no tangential acceleration $(\varepsilon = 0)$ where ε is the ratio of the tangential to normal acceleration. The circular paths come out to be a logical result of this assumption. For tangential decelerations, i.e., $\varepsilon \neq 0$, see Pakdemirli (2023b).

5.4. DETERMINING THE GENERAL FORM FROM A GIVEN SYMMETRY

In the preceding sections, the equation was given and the solution is constructed from the symmetry admitted by the equation. In this section, the reverse problem is addressed. Given the symmetry, what is the general form of the equation that possesses such a symmetry. Only first and second order differential equations will be considered. Indeed in Section 2.6, the problem is addressed for translational and scaling symmetries. A more general approach that can be employed for any type of symmetry will be outlined in this section. The alternative ways to determine the general form from a given symmetry are the differential invariant method and the canonical coordinates.

First Order Differential Equations

Problem 5.16. For the rotational symmetry $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ find the general form of the first order differential equation admitting the symmetry.

Solution

Canonical coordinates will be employed. Hence

$$Xr = -y\frac{\partial r}{\partial x} + x\frac{\partial r}{\partial y} = 0 \quad \rightarrow \quad \frac{dx}{-y} = \frac{dy}{x} = \frac{dr}{0}\,. \tag{5.165}$$

From the first two equations $xdx + ydy = 0$, $x^2 + y^2 = c_1$ and from the last two equations $dr = 0$, $r = c_2 = f(c_1)$ and taking the arbitrary function as the square root function and eliminating the constants,

$$r = \sqrt{x^2 + y^2}\,. \tag{5.166}$$

The second canonical coordinate is calculated next

$$Xs = -y\frac{\partial s}{\partial x} + x\frac{\partial s}{\partial y} = 1 \quad \rightarrow \quad \frac{dx}{-y} = \frac{dy}{x} = \frac{ds}{1}\,. \tag{5.167}$$

From the last two equations

$$ds = \frac{dy}{x} = \frac{dy}{\sqrt{r^2-y^2}} = \frac{1}{r}\frac{dy}{\sqrt{1-\left(\frac{y}{r}\right)^2}} \tag{5.168}$$

or

$$s = \text{Arcsin}\left(\frac{y}{r}\right). \tag{5.169}$$

Indeed the couple (r, s) represents the polar coordinates as canonical coordinates which is an expected result for rotational symmetry.

Calculate now the derivative

$$\frac{ds}{dr} = \frac{s_x + s_y y'}{r_x + r_y y'}\,. \tag{5.170}$$

Since $r_x = x/r$, $r_y = y/r$, $s_x = -y/r^2$, $s_y = x/r^2$, and $\frac{ds}{dr}$ is a function of r only, substituting above

$$\frac{ds}{dr} = G(r) = \frac{-y + xy'}{r(x + yy')}\,. \tag{5.171}$$

Defining a new function $F(r) = rG(r)$ and solving for y', the general form of the first order equation admitting a rotational symmetry is

$$y' = \frac{y + xF(\sqrt{x^2+y^2})}{x - yF(\sqrt{x^2+y^2})}\,. \tag{5.172}$$

Many of the special symmetries and the corresponding form of the differential equations admitting such symmetries are tabulated (Ibragimov, 1999; Cohen, 1911). In Table 5.1, several such symmetries and the corresponding general forms of the differential equations are given.

Table 5.1. General Form of First Order ODEs Admitting a Special Group

Symmetry	Equation
$\frac{\partial}{\partial x}$	$y' = F(y)$
$\frac{\partial}{\partial y}$	$y' = F(x)$
$a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}$	$y' = F(bx + ay)$
$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$	$y' = \frac{y + xF(\sqrt{x^2+y^2})}{x - yF(\sqrt{x^2+y^2})}$
$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$y' = F(y/x)$
$x\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}$	$y' = x^{a-1}F(y/x^a)$
$x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$	$y' = x^{-1}F(xe^{-y})$
$\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$y' = yF(ye^{-x})$
$\frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y}$	$y' = \frac{y}{x} + xF\left(\frac{y}{x}\right)$
$x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$y' = \frac{y}{x} + \frac{1}{x}F\left(\frac{y}{x}\right)$
$xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$	$y' = \frac{y}{x + F(y/x)}$
$y\frac{\partial}{\partial x}$	$y' = \frac{y}{x + F(y)}$
$x\frac{\partial}{\partial y}$	$y' = \frac{y + F(x)}{x}$
$xy\frac{\partial}{\partial x}$	$y' = \frac{y}{x(\ln x + F(y))}$
$xy\frac{\partial}{\partial y}$	$y' = \frac{y(\ln y + F(x))}{x}$
$y\frac{\partial}{\partial y}$	$y' = yF(x)$

Second Order Differential Equations

Problem 5.17. For the uniform scaling symmetry $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ find the general form of the second order differential equation admitting the symmetry.

Solution

This time, the alternative to canonical coordinates, namely the differential invariants method will be employed. The characteristic equation is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy_1}{\eta^{(1)}} = \frac{dy_2}{\eta^{(2)}} \ . \tag{5.173}$$

Substituting $\xi = x$, $\eta = y$, $\eta^{(1)} = 0$ and $\eta^{(2)} = -y_2$ (See Problem 4.2 for calculation of the extended infinitesimals for a scaling symmetry), the characteristic equation is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy_1}{0} = \frac{dy_2}{-y_2} \ . \tag{5.174}$$

Solving the above equations, the invariants are

$$u = \frac{y}{x} \ , \ v = y_1 \ , \ w = xy_2 \ . \tag{5.175}$$

Since the arbitrary constants can be related to each other through an arbitrary function $w = F(u, v)$, then the general form of the equation turns out to be

$$y'' = \frac{1}{x} F\left(\frac{y}{x}, y'\right) . \tag{5.176}$$

Table 5.2 lists a number of special symmetries and the corresponding general forms of the second order equations (Ibragimov, 1999; Cohen, 1911).

Table 5.2. General Forms of Second Order ODEs Admitting a Special Group

Symmetry	Equation
$\frac{\partial}{\partial x}$	$y'' = F(y, y')$
$\frac{\partial}{\partial y}$	$y'' = F(x, y')$
$a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}$	$y'' = F(bx + ay, y')$
$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$	$y'' = (1 + y'^2)^{3/2} F\left(\sqrt{x^2 + y^2}, \frac{y - xy'}{x + yy'}\right)$
$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$y'' = \frac{1}{x} F\left(\frac{y}{x}, y'\right)$
$x\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}$	$y'' = x^{a-2} F(x^{-a}y, x^{1-a}y')$
$\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$y'' = yF(ye^{-x}, y'/y)$
$x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$y'' = \frac{1}{x^3} F\left(\frac{y}{x}, y - xy'\right)$
$xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$	$y'' = \frac{y'^3}{x^3} F\left(\frac{y}{x}, \frac{y - xy'}{y'}\right)$
$y\frac{\partial}{\partial x}$	$y'' = y' F\left(y, \frac{y - xy'}{y'}\right)$
$x\frac{\partial}{\partial y}$	$y'' = F(x, xy' - y)$
$xy\frac{\partial}{\partial x}$	$y'' = -\frac{y'}{x} + xy'^3 F\left(y, \frac{y}{xy'} - \ln x\right)$
$xy\frac{\partial}{\partial y}$	$y'' = \frac{y'^2}{y} + yF\left(x, \frac{xy'}{y} - \ln y\right)$
$y\frac{\partial}{\partial y}$	$y'' = yF\left(x, \frac{y'}{y}\right)$

The classifications get more and more involved for higher order equations and therefore skipped.

5.5. EXERCISES

E.5.1. Consider the first order differential equation

$$x^2 y' + y = 0 .$$

Using the scaling symmetry, determine the integrating factor for the equation and solve the equation.

E.5.2. Consider the linear first order differential equation

$$y' + a(x)y = 0 .$$

Using the symmetry $X = y\frac{\partial}{\partial y}$, determine the integrating factor for the equation and solve the equation.

E.5.3. Find a nontrivial group invariant solution of the equation

$$x^3 y' - 4y^3 = 0\ ,$$

by using the scaling symmetry of the equation.

E.5.4. Reduce the order of the equation and solve

$$y' - \left(\frac{y}{x}\right)^2 = 0\ ,$$

by using canonical coordinates for the symmetry generator $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

E.5.5. Reduce the order of the equation and solve

$$xy' = y + \cot\left(\frac{y}{x}\right)\ ,$$

by using canonical coordinates for the symmetry generator $X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$.

E.5.6. Using the given symmetry, map the solution found in Exercise 5.5 into another solution.

E.5.7. Consider the differential equation describing the path of the object subject to minimum drag force

$$y'' - \alpha(1 + y'^2) = 0\ .$$

Among the eight finite parameter symmetries, infinitesimals corresponding to three of the symmetries of the equation were (Pakdemirli and Aksoy, 2010)

$$\xi = a,\ \ \eta = (b\cos\alpha x + c\sin\alpha x)e^{\alpha y}\ \ .$$

Construct a group invariant solution using the above infinitesimals.

E.5.8. Consider the nonlinear second order differential equation

$$y'' + y'^2 = 0\ .$$

All symmetries were already calculated in Problem 4.6. Calculate a group invariant solution corresponding to the symmetries $X_3 = x^2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, $X_7 = \frac{\partial}{\partial y}$.

E.5.9. Using canonical coordinates, reduce the order of the variable coefficient second order ordinary differential equation (Stephani, 1989)

$$y'' = (x - y)y'^3,$$

by using the symmetry $X = (x - y)\frac{\partial}{\partial x}$.

E.5.10. Using canonical coordinates, reduce the order of the second order ordinary differential equation

$$y'' = \sin y - \ln y' ,$$

by using the symmetry $X = \frac{\partial}{\partial x}$.

E.5.11. Reconsider the problem in E.5.10. Reduce the order by using differential invariants.

E.5.12. Consider the third order nonlinear equation

$$y''' + yy'^2 = 0 .$$

The equation admits the non-uniform scaling with infinitesimals

$$\xi = -2ax , \quad \eta = ay \ .$$

Find a group invariant solution corresponding to the above symmetry.

E.5.13. Consider the curve equation for constant normal acceleration and constant tangential deceleration

$$(1 + y'^2)y''' - (3y' + 2\varepsilon)y''^2 = 0 ,$$

where ε is the ratio of tangential to normal accelerations. The equation admits 4 parameter Lie group of transformations with base generators (Pakdemirli, 2023b)

$$X_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x} ,\ X_3 = \frac{\partial}{\partial y} ,\ X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} .$$

Find a group invariant solution corresponding to the X_2, X_3 and X_4 base generators.

E.5.14. Consider the nonlinear third order equation

$$y''' + yy'^2 = 0 \ .$$

The Lie algebra for the equation is

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = -2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \ .$$

Using both generators, reduce the order of the equation by 2 using differential invariants method.

E.5.15. For the symmetry $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ find the general form of the first order differential equation admitting the symmetry.

E.5.16. For the symmetry $X = y\frac{\partial}{\partial y}$ find the general form of the second order differential equation admitting the symmetry.