CHAPTER 4

ORDINARY DIFFERENTIAL EQUATIONS: SYMMETRY GENERATORS

In this chapter, ordinary differential equations will be treated. There are two basic steps in finding solutions for ordinary differential equations via symmetry methods: 1) Calculate the symmetry generators 2) Use the generators to construct solutions. The first part, namely calculating the symmetries of the ODEs will be outlined in this chapter and the next chapter is devoted to constructing solutions from the generators. First order, second order and higher order differential equations will be treated separately.

4.1. EXTENDED GROUPS AND INFINITESIMAL GENERATORS

A general nonlinear ordinary differential equation of k'th order

$$F\left(x, y, y', y'', \dots, y^{(k)}\right) = 0 \ , \tag{4.1}$$

can be considered as an algebraic equation if the derivatives are redefined as higher order variables such as

$$y_1 = y', \ y_2 = y'', \dots . \quad y_k = y^{(k)} \ , \tag{4.2}$$

and the equation transforms into

$$F(x, y, y_1, y_2, \dots, y_k) = 0 \ . \tag{4.3}$$

In writing the Lie group of transformations for the ODE, one needs to write the transformations for the derivatives also. A sample transformation will look like

$$x^* = x^*(x, y, \epsilon) \ , \tag{4.4}$$
$$y^* = y^*(x, y, \epsilon) \ , \tag{4.5}$$
$$y_1^* = y_1^*(x, y, y_1, \epsilon) \ , \tag{4.6}$$
$$y_2^* = y_2^*(x, y, y_1, y_2, \epsilon) \ , \tag{4.7}$$
$$\vdots$$
$$y_k^* = y_k^*(x, y, y_1, y_2, \dots y_k, \epsilon) \ . \tag{4.8}$$

The transformations of higher order variables cannot be arbitrary, but dictated with the transformations (4.4) and (4.5) via the differentiation operation. For example, in calculating the transformation for first order derivatives

$$y_1^* = \frac{Dy^*}{Dx^*} = \frac{\frac{\partial y^*}{\partial x}dx + \frac{\partial y^*}{\partial y}dy}{\frac{\partial x^*}{\partial x}dx + \frac{\partial x^*}{\partial y}dy} = \frac{\frac{\partial y^*}{\partial x} + \frac{\partial y^*}{\partial y}y_1}{\frac{\partial x^*}{\partial x} + \frac{\partial x^*}{\partial y}y_1} \quad , \tag{4.9}$$

where

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} \, . \tag{4.10}$$

Similarly, for the k'th derivative

$$y_k^* = \frac{Dy_{k-1}^*}{Dx^*} = \frac{\frac{\partial y_{k-1}^*}{\partial x} + \frac{\partial y_{k-1}^*}{\partial y}y_1 + \frac{\partial y_{k-1}^*}{\partial y_1}y_2 + \cdots + \frac{\partial y_{k-1}^*}{\partial y_{k-1}}y_k}{\frac{\partial x^*}{\partial x} + \frac{\partial x^*}{\partial y}y_1} \quad , \tag{4.11}$$

where the total differential now becomes

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \cdots + y_k \frac{\partial}{\partial y_{k-1}} \quad . \tag{4.12}$$

Hence (4.11) gives the transformations for the extended variables for any k with $y_0^* = y^*$for $k = 1$.

In the case of infinitesimal transformations which are the vital elements in calculating symmetries, the transformations are

$$x^* = x + \epsilon\xi(x, y) \, , \tag{4.13}$$

$$y^* = y + \epsilon\eta(x, y) \, , \tag{4.14}$$

$$y_1^* = y_1 + \epsilon\eta^{(1)}(x, y, y_1) \, , \tag{4.15}$$

$$y_2^* = y_2 + \epsilon\eta^{(2)}(x, y, y_1, y_2) \, , \tag{4.16}$$

$$\vdots$$

$$y_k^* = y_k + \epsilon\eta^{(k)}(x, y, y_1, y_2, \ldots y_k) \, . \tag{4.17}$$

The extended infinitesimals $\eta^{(i)}, i = 1,2, .. k$ cannot be independent of ξ and η since the total derivative operator dictates the form of them. To calculate the infinitesimal corresponding to y_k^*, substitute

$$y_{k-1}^* = y_{k-1} + \epsilon\eta^{(k-1)}(x, y, y_1, y_2, \ldots y_{k-1}) \, , \tag{4.18}$$

and (4.13) into (4.11)

$$y_k^* = \frac{D\left(y_{k-1}+\epsilon\eta^{(k-1)}(x,y,y_1,y_2,\ldots y_{k-1})\right)}{D(x+\epsilon\xi(x,y))} = \frac{y_k+\epsilon D\eta^{(k-1)}}{1+\epsilon D\xi} \quad . \tag{4.19}$$

Expand the denominator in a Taylor series up to $O(\epsilon)$

$$y_k^* = \left(y_k + \epsilon D\eta^{(k-1)}\right)(1 - \epsilon D\xi) \; . \tag{4.20}$$

Perform the multiplication keeping up to $O(\epsilon)$ terms

$$y_k^* = y_k + \epsilon\left(D\eta^{(k-1)} - y_k D\xi\right) . \tag{4.21}$$

Comparing (4.21) with (4.17), one concludes that

$$\eta^{(k)} = D\eta^{(k-1)} - y_k D\xi \quad , \quad k = 1,2,\ldots \; . \tag{4.22}$$

Therefore, all extended infinitesimals can be calculated via the recursion relation (4.22) with the total derivative operator defined in (4.12). In open form, the first three extensions are

$$\eta^{(1)} = \eta_x + \left(\eta_y - \xi_x\right)y_1 - \xi_y y_1^2 \; , \tag{4.23}$$

$$\begin{aligned}\eta^{(2)} = {} & \eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)y_1 + (\eta_{yy} - 2\xi_{xy})y_1^2 - \xi_{yy}y_1^3 \\ & +\left(\eta_y - 2\xi_x\right)y_2 - 3\xi_y y_1 y_2 \; ,\end{aligned} \tag{4.24}$$

$$\begin{aligned}\eta^{(3)} = {} & \eta_{xxx} + \left(3\eta_{xxy} - \xi_{xxx}\right)y_1 + 3(\eta_{xyy} - \xi_{xxy})y_1^2 + (\eta_{yyy} - 3\xi_{xyy})y_1^3 \\ & -\xi_{yyy}y_1^4 + 3\left(\eta_{xy} - \xi_{xx}\right)y_2 + 3(\eta_{yy} - 3\xi_{xy})y_1 y_2 - 6\xi_{yy}y_1^2 y_2 \\ & -3\xi_y y_2^2 + \left(\eta_y - 3\xi_x\right)y_3 - 4\xi_y y_1 y_3 \; .\end{aligned} \tag{4.25}$$

The extended infinitesimal generator up to k’th order would then be

$$X^{(k)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y_1} + \eta^{(2)}\frac{\partial}{\partial y_2} + \cdots + \eta^{(k)}\frac{\partial}{\partial y_k} \quad . \tag{4.26}$$

Note that the infinitesimals can be considered as polynomials in terms of the higher order variables y_k with coefficients of the polynomials being linear in terms of ξ and η.

Problem 4.1. Determine the extended group, infinitesimals and generators for the two parameter translational group

$$x^* = x + \epsilon a \ , \tag{4.27}$$
$$y^* = y + \epsilon b \ . \tag{4.28}$$

Solution

Since the infinitesimals are all constants, i.e., $\xi = a$ and $\eta = b$, all derivatives vanish. $\eta^{(k)} = 0$ in (4.23)-(4.25). Hence the extended group elements are $y_k^* = y_k$ and the extended generator to arbitrary orders is $X^{(k)} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$.

Problem 4.2. Determine the extended group, infinitesimals and generators for the two parameter scaling group

$$x^* = e^{\epsilon a}x \ , \tag{4.29}$$
$$y^* = e^{\epsilon b}y \ \ . \tag{4.30}$$

Solution

Expanding the group up to first order

$$x^* = e^{\epsilon a}x \cong (1 + \epsilon a)x \cong x + \epsilon ax \ , \tag{4.31}$$
$$y^* = e^{\epsilon b}y \cong (1 + \epsilon b)y \cong y + \epsilon by \ , \tag{4.32}$$

the infinitesimals are $\xi = ax$ and $\eta = by$. The only nonzero elements in the infinitesimals are $\xi_x = a$ and $\eta_y = b$. Substituting into (4.23)-(4.25),

$$\eta^{(1)} = (b - a)y_1 \ , \eta^{(2)} = (b - 2a)y_2 \ \ , \quad \eta^{(3)} = (b - 3a)y_3 \ \ . \tag{4.33}$$

If the calculations are further carried on via the recursion relation (4.22)

$$\eta^{(k)} = (b - ka)y_k \ , \qquad k = 1,2, \ldots \quad . \tag{4.34}$$

Hence, the extended infinitesimal group and the generator are

$$y_k^* = y_k + \epsilon(b - ka)y_k \ , \qquad k = 1,2, \ldots \tag{4.35}$$

$$X^{(k)} = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + (b - a)y_1\frac{\partial}{\partial y_1} + (b - 2a)y_2\frac{\partial}{\partial y_2} + \cdots + (b - ka)y_k\frac{\partial}{\partial y_k} \tag{4.36}$$

Problem 4.3. Determine the extended group, infinitesimals and generators up to third order for the one parameter rotation group

$$x^* = x\cos\epsilon - y\sin\epsilon \ , \tag{4.37}$$

$$y^* = x\sin\epsilon + y\cos\epsilon\ . \tag{4.38}$$

Solution

Expanding the group in Taylor series and keeping terms up to first order

$$x^* = x\cos\epsilon - y\sin\epsilon \cong \left(1-\frac{\epsilon^2}{2!}\right)x - \left(\epsilon-\frac{\epsilon^3}{3!}\right)y \cong x - \epsilon y\ , \tag{4.39}$$

$$y^* = x\sin\epsilon + y\cos\epsilon \cong \left(\epsilon-\frac{\epsilon^3}{3!}\right)x + \left(1-\frac{\epsilon^2}{2!}\right)y \cong y + \epsilon x\ , \tag{4.40}$$

the infinitesimals are $\xi = -y$ and $\eta = x$. The only nonzero elements in the extended infinitesimals are $\xi_y = -1$ and $\eta_x = 1$. Substituting into (4.23)-(4.25),

$$\eta^{(1)} = 1 + y_1^2\ ,\ \eta^{(2)} = 3y_1y_2\ \ ,\quad \eta^{(3)} = 3y_2^2 + 4y_1y_3\quad . \tag{4.41}$$

Calculations for higher orders can be carried out via the recursion relation (4.22) which is not asked in the problem. A general rule cannot be derived for arbitrary orders and the calculations get more and more involved for higher order extensions. Symbolic manipulation programs are needed for extremely high orders. The infinitesimal group and the extended generator are

$$x^* = x - \epsilon y\ , \tag{4.42}$$

$$y^* = y + \epsilon x\ , \tag{4.43}$$

$$y_1^* = y_1 + \epsilon(1 + y_1^2)\ , \tag{4.44}$$

$$y_2^* = y_2 + \epsilon 3y_1y_2\ , \tag{4.45}$$

$$y_3^* = y_3 + \epsilon(3y_2^2 + 4y_1y_3)\ , \tag{4.46}$$

$$X^{(3)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1+y_1^2)\frac{\partial}{\partial y_1} + 3y_1y_2\frac{\partial}{\partial y_2} + (3y_2^2 + 4y_1y_3)\frac{\partial}{\partial y_3}\ . \tag{4.47}$$

4.2. INVARIANCE OF A DIFFERENTIAL EQUATION

In this section, the invariance concept discussed in the previous chapter for algebraic equations is extended to differential equations. In terms of the new variables defined for derivatives, the differential equation can also be considered as an algebraic equation. In the hypothetical k+2 dimensional space with coordinates $x, y, y_1, y_2, \dots y_k$, the differential equation $F(x, y, y_1, y_2, \dots y_k) = 0$ can be considered as a surface the solution of which is a curve projected into the space of x and y. First the definition of invariance for a differential equation is given.

Definition 4.1. The differential equation $F(x, y, y_1, y_2, \dots y_k) = 0$ is invariant under the extended Lie group of transformations (4.4)-(4.8) and (4.11) if and only if

$$F(x^*, y^*, y_1^*, y_2^*, \dots y_k^*) = 0 \text{ when } F(x, y, y_1, y_2, \dots y_k) = 0 \tag{4.48}$$

The relevant theorem for calculating the symmetries immediately follow from the definition:

Theorem 4.1. If the differential equation $F(x, y, y_1, y_2, \dots y_k) = 0$ is invariant under the given Lie Group of transformations (4.4)-(4.8) and (4.11), then

$$X^{(k)}F \equiv 0 \text{ when } F = 0 \quad , \tag{4.49}$$

where $X^{(k)}$ is the extended generator of the group admitted by the equation

The proof has the essential steps of the proof of Theorem 3.1 given for invariance of algebraic equations and therefore skipped.

4.3. FIRST ORDER DIFFERENTIAL EQUATIONS

Calculating the symmetries of first order differential equations is somewhat different in nature than the higher order differential equations. Assume that the first order differential equation $F(x, y, y_1) = 0$ is solvable in terms of y_1

$$y_1 = f(x, y) \; . \tag{4.50}$$

Applying the invariance condition (4.49) to (4.50)

$$X^{(1)}\big(y_1 - f(x, y)\big) = 0 \; , \tag{4.51}$$

with $X^{(1)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^{(1)} \frac{\partial}{\partial y_1}$

$$\eta^{(1)} - \xi f_x - \eta f_y = 0 \quad , \tag{4.52}$$

or substituting $\eta^{(1)}$ from (4.23) and using (4.50), one has

$$\eta_x + \big(\eta_y - \xi_x\big)f - \xi_y f^2 - \xi f_x - \eta f_y = 0 \; . \tag{4.53}$$

There are two unknowns ξ and η and only one equation, so the above equation is not solvable in general. However, if one specifies ξ, then a special η can be determined from the equation. It can be verified by direct substitution that a trivial solution for the equation is

$$\eta(x,y) = \xi(x,y) f(x,y) \,, \tag{4.54}$$

which does not give much information about the symmetry solutions as will be outlined in the next chapter. Equation (4.53) is called the determining equation for the first order differential equations given by (4.50). For higher orders, the determining equations are polynomials in terms of the higher order variables and can be separated into a number of equations which usually constitutes an over-determined system of linear equations in terms of the variables ξ and η, and hence can be solved with ease.

Problem 4.4. Consider the nonlinear first order differential equation

$$y' + y^2 = 0 \,. \tag{4.55}$$

Find a special symmetry generator of the equation by assuming $\xi = 1$.

Solution

Substitute $f = -y^2$ and $\xi = 1$ into the determining equation (4.53)

$$\eta_x - \eta_y y^2 + 2\eta y = 0 \,. \tag{4.56}$$

Since a special solution is needed, one may assume $\eta = \eta(y)$ which results in

$$-\eta_y y^2 + 2\eta y = 0 \,, \tag{4.57}$$

with a solution $\eta = y^2$. The extended infinitesimal is found by substituting $\xi = 1$ and $\eta = y^2$ into (4.23)

$$\eta^{(1)} = 2yy_1 = -2y^3 \,, \tag{4.58}$$

since $y_1 = -y^2$ from the original equation. The special extended infinitesimal generator for the problem is then

$$X^{(1)} = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} - 2y^3 \frac{\partial}{\partial y_1} \,. \tag{4.59}$$

One may always check the result by applying the generator directly to the equation

$$X^{(1)}(y_1+y^2)=\left(\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}-2y^3\frac{\partial}{\partial y_1}\right)(y_1+y^2)=2y^3-2y^3=0\ , \quad (4.60)$$

which leaves the equation invariant. In fact, for any arbitrary $\xi(x,y)$ function, $\eta(x,y)=-\xi(x,y)y^2$ from (4.54) and a more general special generator for the equation is

$$X=\xi(x,y)\frac{\partial}{\partial x}-\xi(x,y)y^2\frac{\partial}{\partial y}\ . \quad (4.61)$$

The use of generators in constructing solutions is delayed to the next chapter.

Problem 4.5. Consider the nonlinear first order differential equation

$$y'-\sin\left(\frac{y}{x}\right)=0\ . \quad (4.62)$$

Find a special symmetry generator of the equation by assuming $\xi=x$ and $\eta=\eta(y)$.

Solution

Substitute $f=\sin\left(\frac{y}{x}\right)$, $\xi=x$ and $\eta=\eta(y)$ into the determining equation (4.53)

$$(\eta'-1)\sin\left(\frac{y}{x}\right)+\frac{y}{x}\cos\left(\frac{y}{x}\right)-\frac{\eta}{x}\cos\left(\frac{y}{x}\right)=0\ . \quad (4.63)$$

with a solution $\eta=y$. The extended infinitesimal is found by substituting $\xi=x$ and $\eta=y$ into (4.23)

$$\eta^{(1)}=0\ . \quad (4.64)$$

The special extended infinitesimal generator for the problem is then

$$X^{(1)}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\ . \quad (4.65)$$

One may always check the result by applying the generator directly to the equation

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(y_1-\sin\left(\frac{y}{x}\right)\right)=\frac{y}{x}\cos\left(\frac{y}{x}\right)-\frac{y}{x}\cos\left(\frac{y}{x}\right)=0\ , \quad (4.66)$$

which leaves the equation invariant. In fact, for any arbitrary $\xi(x,y)$ function, $\eta(x,y)=\xi(x,y)\sin\left(\frac{y}{x}\right)$ from (4.54) and a more general special generator for the equation is

$$X=\xi(x,y)\frac{\partial}{\partial x}+\xi(x,y)\sin\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\,. \tag{4.67}$$

Note that, the special symmetry given in (4.65) is a uniform scaling symmetry. Inspecting the original equation, one may notice that the structural form is compatible with the general form given in Theorem 2.3 for $a=b=1$.

4.4. SECOND ORDER DIFFERENTIAL EQUATIONS

For second order differential equations, the determining equations can be splitted with respect to coefficients of the higher order variables which can be viewed as polynomials in terms of them. This produces over-determined system of linear partial differential equations in terms of ξ and η which can be solved directly. Lie (1893) proved that for a general second order ordinary differential equation, the possible number of finite parameter Lie group symmetries may be 0,1,2,3 or 8 (See also Mahomed and Leach, 1990, Mahomed, 2007). That means, if you calculate 4-7 finite parameter symmetries, then you have something wrong in your calculations because those numbers are impossible to attain. Lie also showed that if a second-order equation admits an eight-dimensional algebra, it is linearizable by a point transformation and equivalent to the simplest equation $y''=0$ (Lie, 1983; Mahomed, 2007).

Problem 4.6. Consider the nonlinear second order differential equation

$$y''+y'^2=0\,. \tag{4.68}$$

Calculate all symmetry generators of the equation.

Solution

The equation is rewritten in terms of the defined variables

$$F(y_1,y_2)=y_2+y_1^2=0\,, \tag{4.69}$$

where $y_1=y'$and $y_2=y''$. The symmetry generator extended up to second order is

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \eta^{(2)}(x,y,y_1,y_2)\frac{\partial}{\partial y_2}\,. \quad (4.70)$$

From Theorem 4.1, $X^{(2)}F \equiv 0$ when $F = 0$. Applying the generator to the equation

$$\eta^{(2)} + 2y_1\eta^{(1)} = 0 \quad \text{when} \quad y_2 = -y_1^2. \quad (4.71)$$

Substituting $\eta^{(1)}$ and $\eta^{(2)}$ from (4.23) and (4.24) into (4.71) and using $y_2 = -y_1^2$

$$\eta_{xx} + \left(2\eta_{xy} - \xi_{xx}\right)y_1 + (\eta_{yy} - 2\xi_{xy})y_1^2 - \xi_{yy}y_1^3 - \left(\eta_y - 2\xi_x\right)y_1^2 \\ +3\xi_y y_1^3 + 2\eta_x y_1 + 2\left(\eta_y - \xi_x\right)y_1^2 - 2\xi_y y_1^3 = 0\,. \quad (4.72)$$

The above equation can be considered as a polynomial equation in terms of y_1. Hence, the coefficients of the polynomial should vanish since the determining equation is identical to zero.

$(\;)$: $\eta_{xx} = 0$, (4.73)

(y_1) : $2\eta_{xy} - \xi_{xx} + 2\eta_x = 0$, (4.74)

(y_1^2) : $\eta_{yy} - 2\xi_{xy} + \eta_y = 0$, (4.75)

(y_1^3) : $-\xi_{yy} + \xi_y = 0$. (4.76)

Contrary to the first order equations, one has 2 unknowns with 4 equations. The system is a solvable linear over-determined partial differential system. From (4.76),

$$\xi = a(x)e^y + b(x)\,, \quad (4.77)$$

and from (4.73)

$$\eta = c(y)x + d(y)\,. \quad (4.78)$$

Substituting (4.77) and (4.78) into (4.74)

$$2c'(y) - a''(x)e^y - b''(x) + 2c(y) = 0\,. \quad (4.79)$$

The equation above has arbitrary functions in x and y which cannot vary independently. Hence, the suitable choice is to select $a''(x)$ and $b''(x)$ as constants

$$a''(x) = 2a_1 \;\rightarrow\; a(x) = a_1 x^2 + a_2 x + a_3 \;, \tag{4.80}$$

$$b''(x) = 2b_1 \;\rightarrow\; b(x) = b_1 x^2 + b_2 x + b_3 \;. \tag{4.81}$$

Equation (4.79) reduces to

$$c'(y) + c(y) = a_1 e^y + b_1 \;, \tag{4.82}$$

with a solution

$$c(y) = c_1 e^{-y} + \frac{a_1}{2} e^y + b_1 \;. \tag{4.83}$$

The last remaining equation to be satisfied is (4.75). Substituting (4.77) and (4.78) into (4.75)

$$c''(y)x + d''(y) - 2a'(x)e^y + c'(y)x + d'(y) = 0 \;, \tag{4.84}$$

and using (4.80), 4(81) and (4.83)

$$(c_1 e^{-y} + \frac{a_1}{2} e^y)x + d''(y) - 2(2a_1 x + a_2)e^y$$

$$+(-c_1 e^{-y} + \frac{a_1}{2} e^y)x \; + d'(y) = 0 \quad , \tag{4.85}$$

which separates with respect to *x*

(x) : $-3a_1 e^y = 0$, (4.86)

( ) : $d''(y) + d'(y) = 2a_2 e^y$. (4.87)

Solving both equations

$$a_1 = 0 \;,\; d(y) = d_1 e^{-y} + d_2 + a_2 e^y \;. \tag{4.88}$$

Hence, the infinitesimals are

$$\xi = (a_2 x + a_3)e^y + b_1 x^2 + b_2 x + b_3 \;, \tag{4.89}$$

$$\eta = (c_1 e^{-y} + b_1)x + d_1 e^{-y} + d_2 + a_2 e^y \;. \tag{4.90}$$

The symmetries possess maximum number of finite parameter Lie group of transformations that can be attained for second order ODEs, that is 8 parameters a_2, a_3, b_1, b_2, b_3, c_1, d_1 and d_2. The base infinitesimal generators corresponding to each parameter are

$$X_1 = \frac{\partial}{\partial x} \text{ (parameter } b_3) \ , \quad X_2 = x\frac{\partial}{\partial x} \text{ (parameter } b_2) \ ,$$
$$X_3 = x^2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \text{ (parameter } b_1) \ , \ X_4 = e^y\frac{\partial}{\partial x} \text{ (parameter } a_3) \ ,$$
$$X_5 = xe^y\frac{\partial}{\partial x} + e^y\frac{\partial}{\partial y} \text{ (parameter } a_2) \ , \ X_6 = xe^{-y}\frac{\partial}{\partial y} \text{ (parameter } c_1) \ , \qquad (4.91)$$
$$X_7 = \frac{\partial}{\partial y} \text{ (parameter } d_2) \ , \quad X_8 = e^{-y}\frac{\partial}{\partial y} \text{ (parameter } d_1) \ .$$

X_1 and X_7 represent translations in x and y respectively and X_2 represents scalings in x.

Problem 4.7. Consider the nonlinear second order differential equation

$$y'' - 2yy' = 0 \ . \qquad (4.92)$$

Calculate all symmetry generators of the equation.

Solution

The equation is rewritten in terms of the higher order defined variables

$$F(y, y_1, y_2) = y_2 - 2yy_1 = 0 \ , \qquad (4.93)$$

where $y_1 = y'$and $y_2 = y''$. The symmetry generator extended up to second order is

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial y_1} + \eta^{(2)}(x,y,y_1,y_2)\frac{\partial}{\partial y_2} \ . \quad (4.94)$$

From Theorem 4.1, $X^{(2)}F \equiv 0$ when $F = 0$. Applying the generator to the equation

$$\eta^{(2)} - 2y\eta^{(1)} - 2y_1\eta = 0 \quad \text{when} \quad y_2 = 2yy_1. \qquad (4.95)$$

Substituting $\eta^{(1)}$ and $\eta^{(2)}$ from (4.23) and (4.24) into (4.95) and using $y_2 = 2yy_1$,

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_1 + (\eta_{yy} - 2\xi_{xy})y_1^2 - \xi_{yy}y_1^3 + (\eta_y - 2\xi_x)2yy_1$$
$$-6\xi_y yy_1^2 - 2y[\eta_x + (\eta_y - \xi_x)y_1 - \xi_y y_1^2] - 2y_1\eta = 0. \qquad (4.96)$$

The above equation can be considered as a polynomial equation in terms of y_1. Hence, the coefficients of the polynomial should vanish since the determining equation is identical to zero.

$$(\ ) : \quad \eta_{xx} - 2y\eta_x = 0 \ , \tag{4.97}$$

$$(y_1) : \quad 2\eta_{xy} - \xi_{xx} - 2y\xi_x - 2\eta = 0 \ , \tag{4.98}$$

$$(y_1^2) : \quad \eta_{yy} - 2\xi_{xy} - 4y\xi_y = 0 \ , \tag{4.99}$$

$$(y_1^3) : \quad \xi_{yy} = 0 \ . \tag{4.100}$$

The system is a solvable linear over-determined partial differential system. From (4.100)

$$\xi = a_1(x)y + a_2(x) \ . \tag{4.101}$$

Substitute into (4.99) and solve for η

$$\eta = a_1'(x)y^2 + \frac{2}{3}a_1(x)y^3 + b_1(x)y + b_2(x) \ . \tag{4.102}$$

Inserting (4.102) into (4.97)

$$a_1'''(x)y^2 + \frac{2}{3}a_1''(x)y^3 + b_1''(x)y + b_2''(x)$$
$$-2y\left(a_1''(x)y^2 + \frac{2}{3}a_1'(x)y^3 + b_1'(x)y + b_2'(x)\right) = 0 \ , \tag{4.103}$$

the equation is a polynomial in terms of the variable y. Equating the coefficient of y^4 to zero

$$(y^4) : \quad -\frac{4}{3}a_1' = 0 \qquad \rightarrow \qquad a_1(x) = a_1 , \tag{4.104}$$

and the remaining equation can be separated with respect to y

$$(y^2) : \quad -2b_1' = 0 \qquad \rightarrow \qquad b_1(x) = b_1 , \tag{4.105}$$

$$(y) : \quad -2b_2' = 0 \qquad \rightarrow \qquad b_2(x) = b_2 . \tag{4.106}$$

To summarize the calculations up to this stage, the infinitesimals are

$$\xi = a_1 y + a_2(x) \ , \tag{4.107}$$

$$\eta = \frac{2}{3}a_1 y^3 + b_1 y + b_2 \ . \tag{4.108}$$

The last equation to be satisfied is (4.98). Substituting the above infinitesimals

$$-a_2'' - 2ya_2' - \frac{4}{3}a_1y^3 - 2b_1y - 2b_2 = 0 \; . \qquad (4.109)$$

Coefficient of y^3 has to vanish, hence

$$a_1 = 0 \; , \qquad (4.110)$$

and separation with respect to *y* yields

$$(y) \; : \; -2a_2' - 2b_1 = 0 \; \rightarrow \quad a_2(x) = -b_1x + c_1 \; , \qquad (4.111)$$

$$(\;) \; : \; -2b_2 = 0 \quad \rightarrow \quad b_2(x) = 0 \; . \qquad (4.112)$$

Hence the infinitesimals are

$$\xi = -b_1x + c_1 \; , \qquad (4.113)$$

$$\eta = b_1y \; . \qquad (4.114)$$

Therefore the equation admits two parameter Lie group of transformations with the generators

$$X_1 = \frac{\partial}{\partial x} \text{(parameter } c_1) \; , \quad X_2 = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \text{(parameter } b_1) \; . \qquad (4.115)$$

X_1 corresponds to translations in *x* and X_2 corresponds to non-uniform scaling symmetry.

4.5. HIGHER ORDER DIFFERENTIAL EQUATIONS

For ODEs with order higher than 2, as in the previous section for second order ODEs, the determining equations can be splitted with respect to coefficients of the higher order variables which enables solution of a system of linear partial differential equations in terms of ξ and η. If the order of the equation is $k, (k \geq 3)$, the possible number of maximum finite parameter Lie group symmetries is at most $k + 4$ (Lie, 1893). More specifically, for a k'th order equation ($k \geq 3$), the number of point symmetries are from zero to one of *k*+1, *k*+2 or *k*+4 (maximum) (Mahomed and Leach, 1990, Mahomed, 2007).

Problem 4.8. Consider the nonlinear third order differential equation

$$y''' = f(y', y'') \; . \qquad (4.116)$$

Calculate all symmetry generators of the equation for arbitrary function of f.

Solution

In fact, since the symmetries are asked for arbitrary f function, the number of symmetries will be minimal. For some specific forms, the symmetries may extend. Such problems will be addressed in group classification chapter. The minimal number of symmetries is called principle Lie algebra in the context of group classifications. In terms of the new variables, the equation is re-written

$$y_3 = f(y_1, y_2) \ . \tag{4.117}$$

where $y_1 = y'$, $y_2 = y''$ and $y_3 = y'''$. The symmetry generator extended up to third order is

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^{(1)} \frac{\partial}{\partial y_1} + \eta^{(2)} \frac{\partial}{\partial y_2} + \eta^{(3)} \frac{\partial}{\partial y_3} \ . \tag{4.118}$$

From Theorem 4.1, $X^{(3)}F \equiv 0$ when $F = 0$. Applying the generator to the equation, the invariance condition is

$$\eta^{(3)} - \eta^{(2)} f_2 - \eta^{(1)} f_1 = 0 \quad \text{when} \quad y_3 = f \ , \tag{4.119}$$

where $f_1 = f_{y_1}$ and $f_2 = f_{y_2}$. Substituting $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(3)}$ from (4.23)-(4.25) into (4.119) and using $y_3 = f$,

$$\begin{gathered}
\eta_{xxx} + \left(3\eta_{xxy} - \xi_{xxx}\right) y_1 + 3(\eta_{xyy} - \xi_{xxy}) y_1^2 + (\eta_{yyy} - 3\xi_{xyy}) y_1^3 \\
-\xi_{yyy} y_1^4 + 3\left(\eta_{xy} - \xi_{xx}\right) y_2 + 3\left(\eta_{yy} - 3\xi_{xy}\right) y_1 y_2 - 6\xi_{yy} y_1^2 y_2 \\
-3\xi_y y_2^2 + \left(\eta_y - 3\xi_x\right) f - 4\xi_y y_1 f - \eta_{xx} f_2 - \left(2\eta_{xy} - \xi_{xx}\right) y_1 f_2 \\
-(\eta_{yy} - 2\xi_{xy}) y_1^2 f_2 + \xi_{yy} y_1^3 f_2 - \left(\eta_y - 2\xi_x\right) y_2 f_2 + 3\xi_y y_1 y_2 f_2 \\
-\eta_x f_1 - \left(\eta_y - \xi_x\right) y_1 f_1 + \xi_y y_1^2 f_1 = 0 \ .
\end{gathered} \tag{4.120}$$

The above equation block is separated with respect to orders of higher order variables

$$\begin{aligned}
&(\ ) &&: \quad \eta_{xxx} = 0 \ , && (4.121)\\
&(y_1) &&: \quad 3\eta_{xxy} - \xi_{xxx} = 0 \ , && (4.122)\\
&(y_1^2) &&: \quad 3(\eta_{xyy} - \xi_{xxy}) = 0 \ , && (4.123)\\
&(y_1^3) &&: \quad \eta_{yyy} - 3\xi_{xyy} = 0 \ , && (4.124)\\
&(y_1^4) &&: \quad -\xi_{yyy} = 0 \ , && (4.125)\\
&(y_2) &&: \quad 3(\eta_{xy} - \xi_{xx}) = 0 \ , && (4.126)
\end{aligned}$$

$(y_1 y_2):\quad 3(\eta_{yy} - 3\xi_{xy}) = 0\,,$ (4.127)
$(y_1^2 y_2):\quad -6\xi_{yy} = 0\,,$ (4.128)
$(y_2^2):\quad -3\xi_y = 0\,,$ (4.129)
$(f)\quad:\quad \eta_y - 3\xi_x = 0\,,$ (4.130)
$(y_1 f):\quad -4\xi_y = 0\,,$ (4.131)
$(f_2)\quad:\quad -\eta_{xx} = 0\,,$ (4.132)
$(y_1 f_2):\quad -(2\eta_{xy} - \xi_{xx}) = 0\,,$ (4.133)
$(y_1^2 f_2):\quad -(\eta_{yy} - 2\xi_{xy}) = 0\,,$ (4.134)
$(y_1^3 f_2):\quad \xi_{yy} = 0\,,$ (4.135)
$(y_2 f_2):\quad -(\eta_y - 2\xi_x) = 0\,,$ (4.136)
$(y_1 y_2 f_2):\quad 3\xi_y = 0\,,$ (4.137)
$(f_1)\quad:\quad -\eta_x = 0\,,$ (4.138)
$(y_1 f_1):\quad -(\eta_y - \xi_x) = 0\,,$ (4.139)
$(y_1^2 f_1):\quad \xi_y = 0\,.$ (4.140)

Note that this separation is acceptable only for arbitrary *f* functions. If for example $f = y_1$, then (4.130) and (4.122) combine to form one equation, (4.131) and (4.123) combine to form another equation. Since $f_2 = 0$ for this special choice, then one loses all equations (4.132)-(4.137). The last three equations will also interact with the first three equations since $f_1 = 1$. In such special cases, due to fewer restrictions imposed, the symmetries extend.

Returning back to the original case of arbitrary *f*, from (4.129), $\xi_y = 0$ implies $\xi = \xi(x)$. From (4.138) $\eta = \eta(y)$. Then the over-determined system reduces to

$$\xi_{xxx} = 0\,, \quad (4.141)$$
$$\eta_{yyy} = 0\,, \quad (4.142)$$
$$\xi_{xx} = 0\,, \quad (4.143)$$
$$\eta_{yy} = 0\,, \quad (4.144)$$
$$\eta_y - 3\xi_x = 0\,, \quad (4.145)$$
$$\eta_y - 2\xi_x = 0\,, \quad (4.146)$$
$$\eta_y - \xi_x = 0\,. \quad (4.147)$$

From (4.147), $\eta_y = \xi_x$ and substituting into (4.146) $\xi_x = 0$ and $\eta_y = 0$. All equations are satisfied now with the infinitesimals being

$$\xi = a\,, \qquad \eta = b\,. \tag{4.148}$$

The principal Lie algebra therefore consists of two-parameter Lie group of transformations with base generators being translations in the coordinates

$$X_1 = \frac{\partial}{\partial x}\,\text{(parameter } a)\,, \quad X_2 = \frac{\partial}{\partial y}\,\text{(parameter } b)\,. \tag{4.149}$$

In fact, these two symmetries can be guessed from the beginning since x and y are missing in the original equation.

Before closing the chapter, a useful theorem involving symmetries of variable coefficient linear ODEs of arbitrary order will be given.

Theorem 4.2 For the variable coefficient, k'th order linear ODE,

$$a_k(x)y^{(k)} + a_{k-1}(x)y^{(k-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x) \tag{4.150}$$

if one solution $u = u(x)$ of the homogenous equation is known, then the infinite parameter symmetry generator corresponding to this solution is $X = u(x)\frac{\partial}{\partial y}$

Proof

Since $\xi = 0$ and $\eta = u(x)$, from (4.22)

$$\eta^{(i)} = u^{(i)}\;, \quad i = 1,2,\dots k\,. \tag{4.151}$$

Applying the extended generator

$$X = u\frac{\partial}{\partial y} + u'\frac{\partial}{\partial y_1} + u''\frac{\partial}{\partial y_2} + \cdots + u^{(k)}\frac{\partial}{\partial y_k}\;, \tag{4.152}$$

to the equation written in terms of the extended variables $y_i = y^{(i)}$

$$a_k(x)y_k + a_{k-1}(x)y_{k-1} + \cdots + a_2(x)y_2 + a_1(x)y_1 + a_0(x)y = h(x) \tag{4.153}$$

yields

$$a_k(x)u^{(k)} + a_{k-1}(x)u^{(k-1)} + \cdots + a_2(x)u'' + a_1(x)u' + a_0(x)u = 0\,. \tag{4.154}$$

Hence $u(x)$ is the solution of the homogenous equation and $X = u(x)\frac{\partial}{\partial y}$ is a symmetry generator corresponding to this solution. Note that for the homogenous equation $h(x) = 0$, $X = u(x)\frac{\partial}{\partial y}$ is still the symmetry generator.

In the following chapter, methods of constructing symmetry solutions of ODEs will be discussed.

4.6. EXERCISES

E4.1. Determine the extended group, infinitesimals and generators for the two parameter spiral group

$$x^* = x + \epsilon a \ ,$$
$$y^* = e^{\epsilon b} y \ \ .$$

E4.2. Determine the extended group, infinitesimals and generators for the four parameter combined group

$$x^* = e^{\epsilon a} x + \epsilon b \ ,$$
$$y^* = e^{\epsilon c} y + \epsilon d \ .$$

E4.3. Prove Theorem 4.1.

E4.4. Consider the nonlinear first order differential equation

$$y' - \sin y = 0 \ .$$

Find a special symmetry generator of the equation by assuming $\xi = 1$.

E4.5. Consider the nonlinear first order differential equation

$$y' - x^3 y^2 = 0 \ .$$

Find a special symmetry generator of the equation by assuming $\xi = x$ and $\eta = \eta(y)$.

E4.6. Consider the nonlinear second order differential equation

$$y'' + y^3 = 0 \ .$$

Calculate all symmetry generators of the equation.

E4.7. Consider the nonlinear second order differential equation

$$(1-2y)y''-(1+y'^2)=0\ .$$

Calculate all symmetry generators of the equation.

E4.8. Consider the nonlinear third order differential equation

$$(1+y'^2)y'''-3y'y''^2=0\ .$$

Calculate all symmetry generators of the equation.

E4.9. Consider the nonlinear third order differential equation

$$y'''=f(y')y''^2\ .$$

Calculate all symmetry generators of the equation for arbitrary function of f.