CHAPTER 3

LIE GROUPS, INFINITESIMAL GENERATORS AND LIE ALGEBRAS

In this Chapter, the basics of the Lie groups and the corresponding infinitesimal generators, invariance of a curve, multi-parameter Lie group of transformations and the corresponding Lie algebra will be discussed. The ideas about special groups given in Chapter 2 will be generalized in this section.

3.1. LIE GROUPS OF TRANSFORMATIONS

A one parameter Lie group of transformation for two variables with $y = y(x)$ may be written in the general form

$$x^* = x^*(x, y, \epsilon) \ , \tag{3.1}$$

$$y^* = y^*(x, y, \epsilon) \ , \tag{3.2}$$

with ϵ being the continuous group parameter. To be named as a group, the transformations should be closed under the group action, should hold the associative property. Without loss of generality, one may assume that the identity element of the group is $\epsilon = 0$, that is $x^*(x, y, 0) = x,\ y^*(x, y, 0) = y$ and the inverse element is $-\epsilon$, that is $x^*(x^*, y^*, -\epsilon) = x,\ y^*(x^*, y^*, -\epsilon) = y$. The translational, scaling and spiral groups discussed in the previous chapter are all special cases of the general form given in (3.1) and (3.2).

As an example, a one parameter translational symmetry in both coordinates may be

$$x^* = x + \epsilon \ , \tag{3.3}$$

$$y^* = y - \epsilon \ . \tag{3.4}$$

As will be discussed at the end of the chapter, the more general transformation

$$x^* = x + \epsilon a \ , \tag{3.5}$$

$$y^* = y + \epsilon b \ , \tag{3.6}$$

may be considered as a two parameter Lie groups of transformations if parameters a and b are continuous and independent of each other.

The scaling transformation

$$x^* = e^{\epsilon} x \,, \tag{3.7}$$

$$y^* = e^{3\epsilon} y \,, \tag{3.8}$$

is another example of one parameter Lie group of transformations. However, the more general case

$$x^* = e^{\epsilon a} x \,, \tag{3.9}$$

$$y^* = e^{\epsilon b} y \,, \tag{3.10}$$

is a two parameter Lie group of transformations for arbitrary and continuous a and b parameters. If for example $b = -2a$, then one can speak of a one parameter Lie group of transformations for (3.9) and (3.10) since ϵa can always be defined as the new group parameter.

3.2. INFINITESIMAL TRANSFORMATIONS AND GENERATORS

The first thing that can be done is to expand (3.1) and (3.2) in a Taylor series in terms of the group parameter

$$x^* = x^*(x, y, 0) + \epsilon \left.\frac{\partial x^*}{\partial \epsilon}\right|_{\epsilon=0} + \frac{\epsilon^2}{2} \left.\frac{\partial^2 x^*}{\partial \epsilon^2}\right|_{\epsilon=0} + \cdots \,, \tag{3.11}$$

$$y^* = y^*(x, y, 0) + \epsilon \left.\frac{\partial y^*}{\partial \epsilon}\right|_{\epsilon=0} + \frac{\epsilon^2}{2} \left.\frac{\partial^2 y^*}{\partial \epsilon^2}\right|_{\epsilon=0} + \cdots \,. \tag{3.12}$$

If one defines

$$\xi(x, y) = \left.\frac{\partial x^*}{\partial \epsilon}\right|_{\epsilon=0} , \quad \eta(x, y) = \left.\frac{\partial y^*}{\partial \epsilon}\right|_{\epsilon=0} , \tag{3.13}$$

and bearing in mind that $x^*(x, y, 0) = x, y^*(x, y, 0) = y$, the first order Taylor approximation of the general group relations will be

$$x^* = x + \epsilon \xi(x, y) \,, \tag{3.14}$$

$$y^* = y + \epsilon \eta(x, y) \,. \tag{3.15}$$

The above transformations are called infinitesimal Lie group of transformations and ξ and η are called infinitesimals.

It is most practical to define a differential operator using the infinitesimals

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \ , \qquad (3.16)$$

which can be considered as a tangent vector operator with components being ξ and η. From now on, (3.16) will be called infinitesimal generator since with the aid of the generator, it is possible to retrieve the original group relationships. The infinitesimal generators are the essential components to derive analytical solutions to differential equations.

According to the first fundamental theorem of Lie (See Bluman and Kumei, 1989 for a proof), the original group can be retrieved from the integration of the equations

$$\frac{dx^*}{d\epsilon} = \xi(x^*,y^*) \ , \quad x^* = x \quad \text{at} \ \ \epsilon = 0 \ \ , \qquad (3.17)$$

$$\frac{dy^*}{d\epsilon} = \eta(x^*,y^*) \ \ , \quad y^* = y \quad \text{at} \ \ \epsilon = 0 \, . \qquad (3.18)$$

which are named as Lie equations.

Problem 3.1. For the translational symmetry given below,

$$x^* = x + \epsilon \ , \qquad (3.19)$$

$$y^* = y - 2\epsilon \ , \qquad (3.20)$$

find the infinitesimals and the infinitesimal generator. Using the infinitesimals retrieve the group.

Solution

Since the group is already in the form of an infinitesimal transformation, comparing the group with (3.14) and (3.15), the infinitesimals and the generator are

$$\xi = 1, \quad \eta = -2 \ , \qquad X = \frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} \ . \qquad (3.21)$$

To retrieve the group, integrate the equations with the initial conditions

$$\frac{dx^*}{d\epsilon} = 1 \,, \quad x^* = x \quad \text{at} \;\; \epsilon = 0 \,, \tag{3.22}$$

$$\frac{dy^*}{d\epsilon} = -2 \,, \quad y^* = y \quad \text{at} \;\; \epsilon = 0 \,, \tag{3.23}$$

which yields the original group

$$x^* = x + \epsilon \,, \tag{3.24}$$

$$y^* = y - 2\epsilon \,, \tag{3.25}$$

Problem 3.2. For the scaling symmetry given below,

$$x^* = e^{2\epsilon}x \,, \tag{3.26}$$

$$y^* = e^{-3\epsilon}y \,, \tag{3.27}$$

find the infinitesimals, infinitesimal transformations and the infinitesimal generator. Using the infinitesimals, retrieve the group.

Solution

Expand the group in a Taylor series up to first order in the vicinity of $\epsilon = 0$

$$x^* = x + \epsilon 2x + \cdots \,, \tag{3.28}$$

$$y^* = y - \epsilon 3y + \cdots \,, \tag{3.29}$$

which is the corresponding infinitesimal transformation for the group. The infinitesimals and the generator are

$$\xi = 2x, \quad \eta = -3y \,, \quad X = 2x\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y} \;. \tag{3.30}$$

To retrieve the group, integrate the equations with the initial conditions

$$\frac{dx^*}{d\epsilon} = 2x^* \,, \quad x^* = x \quad \text{at} \;\; \epsilon = 0 \,, \tag{3.31}$$

$$\frac{dy^*}{d\epsilon} = -3y^* \,, \quad y^* = y \quad \text{at} \;\; \epsilon = 0 \,, \tag{3.32}$$

which yields

$$x^* = e^{2\epsilon}x \,, \tag{3.33}$$

$$y^* = e^{-3\epsilon} y \, . \tag{3.34}$$

Problem 3.3. For the rotational symmetry given below,

$$x^* = x\cos\epsilon - y\sin\epsilon \, , \tag{3.35}$$

$$y^* = x\sin\epsilon + y\cos\epsilon \, , \tag{3.36}$$

find the infinitesimals, infinitesimal transformations and the infinitesimal generator. Using the infinitesimals, retrieve the group.

Solution

Expand the group in a Taylor series in the vicinity of $\epsilon = 0$,

$$x^* = x\left(1 - \frac{\epsilon^2}{2!} + \cdots\right) - y(\epsilon - \frac{\epsilon^3}{3!} + \cdots) \, , \tag{3.37}$$

$$y^* = x\left(\epsilon - \frac{\epsilon^3}{3!} + \cdots\right) + y\left(1 - \frac{\epsilon^2}{2!} + \cdots\right) , \tag{3.38}$$

and keeping terms up to O(ϵ), the corresponding infinitesimal transformations are

$$x^* = x - \epsilon y \, , \tag{3.39}$$

$$y^* = y + \epsilon x \, . \tag{3.40}$$

Hence, the infinitesimals and the generator are

$$\xi = -y, \quad \eta = x \, , \quad X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \, . \tag{3.41}$$

To retrieve the group, the equations has to be integrated with the initial conditions

$$\frac{dx^*}{d\epsilon} = -y^* \, , \quad x^* = x \quad \text{at} \quad \epsilon = 0 \, , \tag{3.42}$$

$$\frac{dy^*}{d\epsilon} = x^* \, , \quad y^* = y \quad \text{at} \quad \epsilon = 0 \, . \tag{3.43}$$

Since the equations are coupled, they cannot be solved separately. Differentiate the first equation with respect to the group parameter and substitute into the second equation which yields

$$\frac{d^2x^*}{d\epsilon^2} + x^* = 0 \,, \tag{3.44}$$

with a solution

$$x^* = c_1 \cos\epsilon + c_2 \sin\epsilon \,. \tag{3.45}$$

Substituting this solution into the right hand side of (3.43) and integrating

$$y^* = c_1 \sin\epsilon - c_2 \cos\epsilon \,. \tag{3.46}$$

Applying the initial conditions, $c_1 = x$ and $c_2 = -y$ which resumes the original group

$$x^* = x\cos\epsilon - y\sin\epsilon \,, \tag{3.47}$$

$$y^* = x\sin\epsilon + y\cos\epsilon \;. \tag{3.48}$$

The group (3.1) and (3.2) can be expressed alternatively in terms of their infinitesimal generators. If X is a generator of the group, the equivalent expressions are

$$x^* = x^*(x, y, \epsilon) = e^{\epsilon X} x \,, \tag{3.49}$$

$$y^* = y^*(x, y, \epsilon) = e^{\epsilon X} y \,. \tag{3.50}$$

To see this, expand the exponential term

$$e^{\epsilon X} = 1 + \epsilon X + \frac{\epsilon^2}{2} X^2 + \cdots \,, \tag{3.51}$$

and since $Xx = \xi$ and $Xy = \eta$,

$$x^* = x + \epsilon\xi + \frac{\epsilon^2}{2} X\xi + \cdots \,, \tag{3.52}$$

$$y^* = y + \epsilon\eta + \frac{\epsilon^2}{2} X\eta + \cdots \,. \tag{3.53}$$

The representation (3.49)-(3-51) is the Lie series representation of the group (Bluman and Kumei, 1989). Terms up to $O(\epsilon)$ represent the infinitesimal transformations given in (3.14) and (3.15) and the whole expansion represent the original group in view of (3.11) and (3.12).

Problem 3.4. Retrieve the rotational group from its generator $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ using the Lie series.

Solution

Since $Xx = -y$, $X^2x = X(-y) = -x$, $X^3x = XX^2x = X(-x) = y$, $X^4x = x$, the Lie series

$$x^* = x + \epsilon Xx + \frac{\epsilon^2}{2}X^2x + \frac{\epsilon^3}{3!}X^3x + \cdots , \tag{3.54}$$

yields

$$x^* = x - \epsilon y - \frac{\epsilon^2}{2}x + \frac{\epsilon^3}{3!}y + \frac{\epsilon^4}{4!}x \dots , \tag{3.55}$$

or

$$x^* = x\left(1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!}\right) - y\left(\epsilon - \frac{\epsilon^3}{3!} + \cdots\right) = x\cos\epsilon - y\sin\epsilon . \tag{3.56}$$

A similar calculation retrieves the *y*-coordinate transformation

$$y^* = x\left(\epsilon - \frac{\epsilon^3}{3!} + \cdots\right) + y\left(1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!}\right) = x\sin\epsilon + y\cos\epsilon . \tag{3.57}$$

3.3. INVARIANCE OF A CURVE AND FAMILY OF CURVES

In this section, the invariance of curves and family of curves will be defined and the condition of invariance in terms of the infinitesimal generators will be derived. In the following chapters, the invariance of differential equations will be considered in a similar way.

Definition 3.1. The curve $F(x, y) = 0$ is invariant under the Lie group of transformations (3.1) and (3.2) if and only if

$$F(x^*, y^*) = 0 \text{ when } F(x, y) = 0 \tag{3.58}$$

Corollary 3.1. If $F(x, y)$ is infinitely differentiable, then for a Lie group of transformation with the corresponding infinitesimal generator being X

$$F(x^*, y^*) = F(e^{\epsilon X}x, e^{\epsilon X}y) = e^{\epsilon X}F(x, y) \tag{3.59}$$

Proof

Expand $F(e^{\epsilon X}x, e^{\epsilon X}y)$ in the vicinity of $\epsilon = 0$ in a Taylor series

$$F(e^{\epsilon X}x, e^{\epsilon X}y) = F(x, y) + \epsilon \frac{\partial F}{\partial x^*}\frac{\partial x^*}{\partial \epsilon}\bigg|_{\epsilon=0} + \epsilon \frac{\partial F}{\partial y^*}\frac{\partial y^*}{\partial \epsilon}\bigg|_{\epsilon=0} + O(\epsilon^2) ,$$

$$= F(x, y) + \epsilon\xi(x, y)\frac{\partial F}{\partial x} + \epsilon\eta(x, y)\frac{\partial F}{\partial y} + O(\epsilon^2) ,$$

$$= F(x, y) + \epsilon XF + O(\epsilon^2) ,$$

$$= \big(1 + \epsilon X + O(\epsilon^2)\big)F = e^{\epsilon X}F(x, y) ,$$

in view of (3.13) and (3.51).

Theorem 3.1. If the curve $F(x, y) = 0$ is invariant under the given Lie Group of transformations (3.1) and (3.2), then

$$XF \equiv 0 \text{ when } F = 0 \quad , \tag{3.60}$$

where X is the generator of the group admitted by the equation

Proof
From Corollary 3.1

$$F(x^*, y^*) = F(e^{\epsilon X}x, e^{\epsilon X}y) = e^{\epsilon X}F(x, y) \ ,$$

or from (3.39) since $e^{\epsilon X} = 1 + \epsilon X + \frac{\epsilon^2}{2}X^2 + \cdots$,

$$F(x^*, y^*) = F(x, y) + \epsilon XF + \frac{\epsilon^2}{2}X^2F + \cdots \ .$$

If $XF = 0$, then $X^2F = X(XF) = X(0) = 0$ and all higher order terms vanish accordingly. Then $F(x^*, y^*) = F(x, y) = 0$ for the algebraic equation.

Problem 3.5. Show that the line $y = mx$ is invariant under uniform scaling.

Solution
Since $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ for uniform scaling $X(y - mx) = y - mx = 0$ in view of Theorem 3.1.

Alternatively, if one uses the analysis of Chapter 2, then substitute $x^* = \lambda^a x$, $y^* = \lambda^b y$ into the line equation $\lambda^{-b}y^* = m\lambda^{-a}x^*$, multiply the equation by λ^b

to obtain $y^* = m\lambda^{b-a}x^*$. For invariance, $b - a = 0$ or $b = a$ which corresponds to uniform scaling.

In the case of family of curves, the invariance definition is slightly altered.

Definition 3.2. The family of curves $F(x,y) = c$ where c is a constant parameter is invariant under the Lie group of transformations (3.1) and (3.2) if and only if

$$F(x^*, y^*) = c^* \qquad \text{when } F(x,y) = c \;, \tag{3.61}$$

for some other constant c^*

Theorem 3.2. If the family of curves $F(x,y) = c$ is invariant under the given Lie group of transformations (3.1) and (3.2), then

$$XF \equiv c^*, \tag{3.62}$$

where c^* can be normalized, i.e. $c^* = 1$

See Bluman and Kumei (1989) for the proof of the theorem. For more than two variables, the ideas for invariant curves can be generalized to invariant surfaces accordingly.

Problem 3.6. Show that the family of circles $x^2 + y^2 = r^2$ is invariant under uniform scaling.

Solution

Since $X = ax\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}$ for uniform scaling $X(x^2 + y^2) = 2ax^2 + 2ay^2 = 2ar^2 = 1$ for $r = \frac{1}{\sqrt{2a}}$ hence admits the uniform scaling.

Applying the group transformation of the previous chapter $x^* = \lambda^a x$, $y^* = \lambda^b y$ into the family of circles $\lambda^{-2a}x^{*2} + \lambda^{-2b}y^{*2} = r^2$, multiplying the equation by λ^{2b} to obtain $\lambda^{2(b-a)}x^{*2} + y^{*2} = \lambda^{2b}r^2$. For invariance, $b = a$ which transforms the equation into $x^{*2} + y^{*2} = r^{*2}$ where $r^* = \lambda^b r$. The transformation $x^* = \lambda^a x$, $y^* = \lambda^a y$ corresponds to uniform scaling.

3.4. MULTI-PARAMETER LIE GROUP OF TRANSFORMATIONS

As mentioned earlier in this chapter and in the previous chapter, the group may contain more than one continuous parameter. The number of independent parameters in a group then constitutes multi-parameter groups. For each distinct parameter, an infinitesimal generator can be written. For an r-parameter Lie group of transformations, there are r independent generators called the base generators of the group. A relationship is defined between the generators.

Definition 3.3. For an r-parameter Lie Group of transformations with X_α and X_β any two infinitesimal generators of the group, the commutator of them is

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha \tag{3.63}$$

The immediate consequence of the definition is

$$[X_\alpha, X_\beta] = -[X_\beta, X_\alpha] \,, \tag{3.64}$$

which shows that the commutator is skew symmetric. The Jacobi identity is also satisfied

$$\left[X_\alpha, [X_\beta, X_\gamma]\right] + \left[X_\beta, [X_\gamma, X_\alpha]\right] + \left[X_\gamma, [X_\alpha, X_\beta]\right] = 0 \,. \tag{3.65}$$

The second fundamental theorem of Lie is stated below:

Theorem 3.3. The commutator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator

$$[X_\alpha, X_\beta] = C_{\alpha\beta}^{\gamma} X_\gamma \,, \tag{3.66}$$

where $C_{\alpha\beta}^{\gamma}$ are called structure constants and $\alpha, \beta, \gamma = 1,2, \dots r$

In accordance with the Einstein summation convention, the repeated indexes represent summation over the index. α and β are the free indexes whereas γ is the repeated index over which summation must be performed.

For a proof of the theorem, see Ibragimov (1999). Equation (3.66) is called the commutation relations. The third fundamental theorem of Lie states the relationship between the structure constants.

Theorem 3.4. The structure constants given in (3.66) satisfy

$$C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha} \tag{3.67}$$

$$C^{\rho}_{\alpha\beta}C^{\delta}_{\rho\gamma} + C^{\rho}_{\beta\gamma}C^{\delta}_{\rho\alpha} + C^{\rho}_{\gamma\alpha}C^{\delta}_{\rho\beta} = 0 \tag{3.68}$$

The above relationships are direct consequences of skew symmetry and Jacobi identity.

Definition 3.4. A Lie algebra $\mathcal{L}^r$ is a vector space of *r* independent base vectors with commutator relation (3.63) satisfying (3.64) and (3.65) and closed under the commutator operation

A Lie algebra may contain subalgebras given by the below definition.

Definition 3.5. A subspace with *s* number of base vectors $\mathcal{L}^s \subset \mathcal{L}^r$ $(s < r)$ is a subalgebra of $\mathcal{L}^r$ if for any $X_\alpha, X_\beta \in \mathcal{L}^s$, $[X_\alpha, X_\beta] \in \mathcal{L}^s$

Special forms of subalgebras prove to be useful in solving higher order ordinary differential equations**.**

Definition 3.6. A subalgebra with *s* number of base vectors $\mathcal{L}^s \subset \mathcal{L}^r$ $(s < r)$ is an ideal or normal subalgebra of $\mathcal{L}^r$ if for any $X_\alpha \in \mathcal{L}^s$, $X_\beta \in \mathcal{L}^r$, $[X_\alpha, X_\beta] \in \mathcal{L}^s$

Next follows the definition of solvable Lie algebras.

Definition 3.7. $\mathcal{L}^s$ is a solvable Lie algebra if there exists a hierarchy of algebras

$$\mathcal{L}^1 \subset \mathcal{L}^2 \subset \cdots \subset \mathcal{L}^{s-2} \subset \mathcal{L}^{s-1} \subset \mathcal{L}^s \tag{3.69}$$

such that $\mathcal{L}^i$ is a Lie algebra and $\mathcal{L}^{i-1}$ is an ideal of $\mathcal{L}^i$, $i = 1,2, \dots s$

Definition 3.8. A Lie algebra $\mathcal{L}^r$ is Abelian if for any $X_\alpha, X_\beta \in \mathcal{L}^r$,

$$[X_\alpha, X_\beta] = 0 \tag{3.70}$$

The following theorems immediately follow.

Theorem 3.5. Every Abelian Lie algebra is a solvable Lie algebra

Theorem 3.6. Every two dimensional Lie algebra is solvable

See Bluman and Kumei (1989) for proofs.

Problem 3.7. For the combined transformation of translational, scaling and spiral groups, construct the Lie algebra and identify the subalgebras

$$x^* = e^{\epsilon a}x + \epsilon b \,, \tag{3.71}$$
$$t^* = e^{\epsilon c}t + \epsilon d \,, \tag{3.72}$$

if a, b, c and d are arbitrary continuous parameters independent of each other.

Solution

The combined group is a 4-parameter Lie group of transformations. The base generators corresponding to each parameter are

$$X_1 = \frac{\partial}{\partial x} \text{ (parameter } b\text{)}, \quad X_2 = \frac{\partial}{\partial t} \text{ (parameter } d\text{)} \,, \tag{3.73}$$

$$X_3 = x\frac{\partial}{\partial x} \text{ (parameter } a\text{)}, \quad X_4 = t\frac{\partial}{\partial t} \text{ (parameter } c\text{)} \,. \tag{3.74}$$

It is more convenient to summarize the commutator relations for the base vectors in Table 3.1.

Table 3.1. Lie Algebra for the Combined Group

	X_1	X_2	X_3	X_4
X_1	0	0	X_1	0
X_2	0	0	0	X_2
X_3	$-X_1$	0	0	0
X_4	0	$-X_2$	0	0

It is obvious that the 4-parameter Lie algebra is closed under the commutator operation. Note that the diagonal elements are always zero and the matrix is skew-symmetric with respect to its diagonal.

A three dimensional subalgebra is formed by X_1, X_2 and X_3. In fact any 3 dimensional basis choices will yield a subalgebra since the operations are closed under it. All two dimensional bases constitute a subalgebra also since they are closed under the action. Some of the 2D subalgebras are special, namely X_1 and X_2 ; X_3 and X_4; X_1 and X_4; X_2 and X_3. They are called Abelian groups since all elements of them are zero. Although X_1 and X_3 constitute a 2D subalgebra, it is not Abelian because of the nonzero elements. Same applies to X_2 and X_4.

Problem 3.8. For the combined transformation of translational and rotation groups, construct the Lie Algebra and identify the subalgebras

$$x^* = x\cos\epsilon a - y\sin\epsilon a + \epsilon b\ , \quad (3.75)$$
$$y^* = x\sin\epsilon a + y\cos\epsilon a + \epsilon c\ , \quad (3.76)$$

if a, b and c are arbitrary continuous parameters independent of each other.

Solution

The combined group is a 3-parameter Lie group of transformations. The base generators corresponding to each parameter are

$$X_1 = \frac{\partial}{\partial x}\ \text{(parameter } b\text{)}, \quad X_2 = \frac{\partial}{\partial y}\ \text{(parameter } c\text{)}\ , \quad (3.77)$$

$$X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\ \text{(parameter } a\text{)}\ . \quad (3.78)$$

It is more convenient to summarize the commutator relations for the base vectors in Table 3.2.

Table 3.2. Lie Algebra for the Combined Rotation and Translations Group

	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	$-X_1$
X_3	$-X_2$	X_1	0

It is obvious that the 3-parameter Lie Algebra is closed under the commutator operation. X_1 and X_2 form a 2D subalgebra which is Abelian. However, X_1 and X_3 ; X_2 and X_3 do not constitute 2D subalgebras since they are not closed. Note that the algebra is solvable since $\mathcal{L}^1 \subset \mathcal{L}^2 \subset \mathcal{L}^3$ where $\mathcal{L}^3$ has base vectors X_1, X_2, X_3 , $\mathcal{L}^2$ has base vectors X_1, X_2 and $\mathcal{L}^1$ has base vector X_1.

Problem 3.9. For the base generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}\ , \quad X_3 = x^2\frac{\partial}{\partial x}\ , \quad (3.79)$$

construct the Lie algebra, identify the subalgebras and determine their properties.

Solution

The elements of the Lie algebra are calculated and given in Table 3.3

Table 3.3. Lie Algebra for the Basis (3.79)

	X_1	X_2	X_3
X_1	0	X_1	$2X_2$
X_2	$-X_1$	0	X_3
X_3	$-2X_2$	$-X_3$	0

It is obvious that the 3-parameter Lie Algebra is closed under the commutator operation. X_1 and X_2 form a 2D subalgebra which is non-Abelian. Similarly, X_2 and X_3 form also a 2D subalgebra which is again non-Abelian. However, X_1 and X_3 do not constitute a 2D subalgebra since it is not closed, i.e. $[X_1, X_3] = 2X_2$. Furthermore, the three dimensional Lie Algebra is not solvable according to Definition 3.7. Either of the subalgebras $\mathcal{L}^2$, i.e. X_1 and X_2 or X_2 and X_3 are not ideals of $\mathcal{L}^3$.

3.5. EXERCISES

E.3.1. For the spiral symmetry given below,

$$x^* = x + \epsilon ,$$

$$y^* = e^{\epsilon} y ,$$

find the infinitesimals and the infinitesimal generator. Using the infinitesimals retrieve the group.

E.3.2. For the combined symmetry given below,

$$x^* = e^{2\epsilon} x + \epsilon ,$$

$$y^* = e^{-\epsilon} y ,$$

find the infinitesimals and the infinitesimal generator. Using the infinitesimals retrieve the group.

E.3.3. Retrieve the non-uniform scaling group from its generator $X = -2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ using the Lie series.

E.3.4. Find the specific form of non-uniform scaling under which the parabola $y = x^2$ remains invariant under the transformation.

E.3.5. Find a such that the infinitesimal generator $X = a\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is a symmetry of the curve $y = e^{-x}$.

E.3.6. Show that the family of ellipses $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is invariant under uniform scaling.

E.3.7. Prove skew-symmetry (3.64) and Jacobi identity (3.65) for commutator operation.

E.3.8. Prove Theorem 3.4.

E.3.9. For the base generators of translations, uniform scaling and rotation, construct the Lie algebra and identify the subalgebras and their properties.

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$