CHAPTER 2

SOLUTIONS BY SPECIAL GROUP TRANSFORMATIONS

The general Lie group theory requires extensive computations and the theory is much involved. In this chapter, as an introduction to the topic, special group transformations admitted by the differential equations will be considered which does not require extensive algebra. In this way, the general ideas behind the symmetry methods can be given in a simple and understandable way before discussing the general theory. Translational symmetries, scaling symmetries, spiral group symmetries are the most common symmetries that are inherited by the differential equations. It is an easy task to check whether such symmetries are admitted by the given equation. If the special transformation is determined, then solutions based on such symmetries can be constructed in a systematic way. Both ordinary and partial differential equations will be considered in this chapter. A unified approach including translational, scaling and spiral symmetries will also be discussed.

In the direct analysis, the equation to be solved is given and the test yields the special transformations admitted by the equation which then produces the so called symmetry solutions. The reverse method is to find the general structural form of the equation for which a given special group of transformation is admitted. This reverse method will also be discussed in this chapter. Another important topic is the group classification problem which arises when the equation contains arbitrary parameters or arbitrary functions. Such cases are investigated under the topic of general Lie group theory. However, in this section, group classification is done with respect to a given special group transformation as a preliminary introduction to the topic. Chapter 2 is self-contained and using the ideas, the applied oriented researchers may start finding solutions to their equations without a general grasp of the Lie Group theory which will follow in the subsequent chapters. For applications of special Lie group of transformations to physical problems, see Pakdemirli and Yürüsoy (1998), Pakdemirli and Şahin (2006a, 2006b), Hansen and Na (1968), Timol and Kalthia (1986), Pakdemirli (1992, 1993, 1994a, 1994b), Kılıç et al. (2004), Hayat et al. (2013), Abbasbandy et al. (2008).

2.1. TRANSLATIONAL SYMMETRIES

Such transformations express continuous transformations in terms of the group parameter along the coordinates. One ordinary and one partial differential equation will be considered. Using the translational symmetries, group-invariant solutions will be presented.

Problem 2.1. Consider the ordinary differential equation

$$y'' + y'^2 = 0 \,, \tag{2.1}$$

where $y = y(x)$. Since the independent and dependent variables are absent in the equation, the translations in both coordinates

$$x^* = x + \epsilon a \,, \tag{2.2}$$
$$y^* = y + \epsilon b \,, \tag{2.3}$$

are admitted by the equation. ϵ is the continuous group transformation parameter and a and b are some constants. To see that the equation admits the transformation, express the equation in terms of the new coordinates

$$\frac{dy}{dx} = \frac{d}{dx^*}(y^* - \epsilon b)\frac{dx^*}{dx} = \frac{dy^*}{dx^*} \;. \tag{2.4}$$

Similarly the second derivative is transformed by the chain rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx^*}\left(\frac{dy}{dx}\right)\frac{dx^*}{dx} = \frac{d^2y^*}{dx^{*2}} \,. \tag{2.5}$$

Note that the transformations of the higher order derivatives are not arbitrary but dictated by the original group and derivation process. Hence

$$y'^* = y' \;, \; y''^* = y'' \;. \tag{2.6}$$

The above derivative transformations will be called extended Lie Group transformations and the general formulas for such extensions will be given in the subsequent chapters. The transformed equation then looks exactly as the same equation given in (2.1)

$$y^{*\prime\prime} + y^{*\prime 2} = 0 \,, \tag{2.7}$$

which leads to the conclusion that Eq. (2.1) is invariant (unchanged) under the transformations (2.2) and (2.3).

To find a group invariant solution, assume the group parameter to be extremely small and define the incremental changes in the coordinates

$$dx = x^* - x = \epsilon a, \qquad dy = y^* - y = \epsilon b \,. \tag{2.8}$$

Eliminating ϵ between the equations, the direct relationship between the variables is dictated by the characteristic equation

$$\frac{dx}{a} = \frac{dy}{b} \,, \tag{2.9}$$

with a solution

$$y = mx + y_0, \quad m = \frac{b}{a} \,. \tag{2.10}$$

The above solution has to satisfy the original equation, which when substituted yields

$$m^2 = 0 \quad \rightarrow \quad y = y_0 \quad . \tag{2.11}$$

Therefore, the constant trivial solution turns out to be a group invariant solution of the problem. In the following chapters, it will be shown that the main role of symmetries of ordinary differential equations will be to reduce the order of the equation rather than determining directly the group invariant solutions.

Next, a partial differential equation is treated.

Problem 2.2. Consider the dimensionless wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad , \tag{2.12}$$

where $u = u(x,t)$. The translations in all coordinates can be written as

$$x^* = x + \epsilon a \,, \tag{2.13}$$
$$t^* = t + \epsilon b \,, \tag{2.14}$$
$$u^* = u + \epsilon c \,, \tag{2.15}$$

for the group parameter ϵ and for some constants a, b and c. Using the chain rule, Eq. (2.12) is transformed into

$$\frac{\partial^2 u^*}{\partial t^{*2}} = \frac{\partial^2 u^*}{\partial x^{*2}} \,. \tag{2.16}$$

Hence, Eqs. (2.13)-(2.15) represent the most general translational symmetries admitted by the wave equation. In other words, the wave equation remains invariant under the given translational symmetries.

Defining the incremental changes in the coordinates

$$dx = x^* - x = \epsilon a, \qquad dt = t^* - t = \epsilon b \ , \quad du = u^* - u = \epsilon c \tag{2.17}$$

and solving for ϵ, the characteristic equation is

$$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{c} \ . \tag{2.18}$$

Since the parameters are arbitrary, one may choose $c = 0$, $a = mb$. There are two independent equations above. Choosing $\frac{dx}{mb} = \frac{dt}{b}$ and solving yields the so called similarity variable

$$\eta = x - mt \ , \tag{2.19}$$

where η is a constant of integration. Next, the equation $\frac{dx}{mb} = \frac{du}{0}$ yields $u = u_0$ another constant which has to be related to the constant η via an arbitrary function $F = F(\eta)$. Hence

$$u = F(\eta) \ . \tag{2.20}$$

Substituting (2.20) and (2.19) into the original equation (2.12)

$$F''(m^2 - 1) = 0 \ . \tag{2.21}$$

One now has two choices; either $F'' = 0$ or $m = \mp 1$. The first choice leads to

$$F = c_1\eta + c_2 \ \rightarrow \ \ u = k_1 x + k_2 t + k_3 \ , \tag{2.22}$$

which is a simple solution that can be found by direct inspection of the original equation. The next choice of $m = \mp 1$ leads to nontrivial solutions

$$u = F_1(x - t) + F_2(x + t) \ , \tag{2.23}$$

which is a superposition of traveling wave solutions to the right and left. Boundary and initial conditions are needed to determine the specific forms of arbitrary functions F_1 and F_2.

The role of the symmetry for the specific partial differential equation is to transform it into an ordinary differential equation. As will be discussed in the subsequent chapters, one parameter Lie Group transformation reduces the n-independent variable PDE into an n-1-independent variable PDE. For two independent variables, the PDE reduces to an ODE as outlined. η and $F(\eta)$ are called the similarity variables or alternatively group invariants. The solutions are called similarity solutions or alternatively group invariant solutions. As will be discussed later, in addition to group invariant solutions, symmetries may also be employed to find another solution from a known solution or may be employed to transfer the equation into a simpler form.

2.2. SCALING SYMMETRIES

Scaling symmetries allow proportional reduction or increase in the coordinates. Again two sample problems, one ordinary and one partial differential equation will be treated. The solutions will be group-invariant solutions.

Problem 2.3. Consider the ordinary differential equation

$$y'' + xy'^2 = 0 , \tag{2.24}$$

where $y = y(x)$. The scaling symmetries can be written as

$$x^* = \lambda^a x , \tag{2.25}$$
$$y^* = \lambda^b y , \tag{2.26}$$

with $\lambda = 1 + \epsilon$. Alternatively the symmetries can be expressed as

$$x^* = e^{\epsilon a} x , \tag{2.27}$$
$$y^* = e^{\epsilon b} y , \tag{2.28}$$

where ϵ is the continuous group transformation parameter. Using chain rule and considering the group transformations (2.25) and (2.26), the first derivative transforms into

$$\frac{dy}{dx} = \frac{d}{dx^*}(\lambda^{-b} y^*)\frac{dx^*}{dx} = \lambda^{a-b}\frac{dy^*}{dx^*} \ . \tag{2.29}$$

Similarly the second derivative is

$$\frac{d^2y}{dx^2} = \frac{d}{dx^*}\left(\frac{dy}{dx}\right)\frac{dx^*}{dx} = \lambda^{2a-b}\frac{d^2y^*}{dx^{*2}} \ . \tag{2.30}$$

Substituting into the original equation

$$\lambda^{2a-b}\frac{d^2y^*}{dx^{*2}} + \lambda^{a-2b}x^*\left(\frac{dy^*}{dx^*}\right)^2 = 0\ , \tag{2.31}$$

and multiplying the equation with λ^{b-2a} yields

$$\frac{d^2y^*}{dx^{*2}} + \lambda^{-a-b}x^*\left(\frac{dy^*}{dx^*}\right)^2 = 0\ . \tag{2.32}$$

Comparing (2.32) with the original equation (2.24), one concludes that $-a-b=0$ for invariance or

$$b = -a\ . \tag{2.33}$$

To find a group invariant solution, assume the group parameter to be extremely small

$$x^* = \lambda^a x = (1+\epsilon)^a x \cong (1+\epsilon a)x = x + \epsilon a x\ , \tag{2.34}$$
$$y^* = \lambda^b y = (1+\epsilon)^b y \cong (1+\epsilon b)y = y + \epsilon b y\ , \tag{2.35}$$

and the incremental changes in the coordinates are

$$dx = x^* - x = \epsilon a x, \qquad dy = y^* - y = \epsilon b y = -\epsilon a y\ , \tag{2.36}$$

in view of (2.33). Eliminating ϵ between the equations, the characteristic equation is

$$\frac{dx}{ax} = \frac{dy}{-ay}\ , \tag{2.37}$$

with a solution

$$y = \frac{c}{x}\ , \tag{2.38}$$

The above solution has to satisfy the original equation, which when substituted yields

$$c = -2\ . \tag{2.39}$$

Therefore, the scaling group invariant solution for the problem is

$$y = -\frac{2}{x}\ . \tag{2.40}$$

As mentioned in Problem 2.1, in the later chapters, the reduction of the order of equation for ODEs will be discussed using symmetries.

Problem 2.4. Consider the dimensionless heat conduction (diffusion) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} , \tag{2.41}$$

where $u = u(x, t)$. The alternative form for scaling transformations can be used

$$x^* = e^{\epsilon a} x , \tag{2.42}$$
$$t^* = e^{\epsilon b} t , \tag{2.43}$$
$$u^* = e^{\epsilon c} u , \tag{2.44}$$

with ϵ being the group parameter and a, b and c some constant parameters. The equation is transformed into

$$e^{\epsilon(b-c)} \frac{\partial u^*}{\partial t^*} = e^{\epsilon(2a-c)} \frac{\partial^2 u^*}{\partial x^{*2}} . \tag{2.45}$$

Multiply the equation by $e^{\epsilon(c-b)}$

$$\frac{\partial u^*}{\partial t^*} = e^{\epsilon(2a-b)} \frac{\partial^2 u^*}{\partial x^{*2}} . \tag{2.46}$$

Comparing the transformed equation with the original equation, the conditions

$$b = 2a , \quad c \text{ arbitrary}, \tag{2.47}$$

are required for the scaling symmetry to be admitted by the heat conduction equation. The associated characteristic equation is

$$\frac{dx}{ax} = \frac{dt}{2at} = \frac{du}{cu} . \tag{2.48}$$

One may choose $c = 0$. Integrating the first two equations

$$\ln x = \frac{1}{2} \ln t + ln\xi , \tag{2.49}$$

where ξ is a constant of integration. Solving for ξ

$$\xi = \frac{x}{\sqrt{t}} . \tag{2.50}$$

Next, solving the first and third equation for $c = 0$, $u = u_0$ another constant, which has to be related to the other constant via an arbitrary function $F = F(\xi)$. Hence

$$u = F(\xi)\ . \tag{2.51}$$

Equations (2.50) and (2.51) are the similarity variable and function for the transformation. When substituted into the original equation, if there is no error involved, the equations must appear only in terms of the new similarity variables. First the derivatives should be calculated

$$\frac{\partial u}{\partial t} = F' \frac{\partial \xi}{\partial t} = -\frac{x}{2t^{3/2}} F'\ , \tag{2.52}$$

$$\frac{\partial u}{\partial x} = F' \frac{\partial \xi}{\partial x} = \frac{1}{t^{1/2}} F'\ , \tag{2.53}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{t^{1/2}} F'' \frac{\partial \xi}{\partial x} = \frac{1}{t} F''. \tag{2.54}$$

Substituting into the original equation

$$-\frac{x}{2t^{3/2}} F' = \frac{1}{t} F'', \tag{2.55}$$

and multiplying the equation by t and rearranging

$$F'' + \frac{\xi}{2} F' = 0\ , \tag{2.56}$$

which is an ODE in terms of the similarity variables. Defining $F' = p$,

$$\frac{dp}{d\xi} = -\frac{\xi}{2} p\ , \tag{2.57}$$

which can be separated $\frac{dp}{p} = -\frac{\xi}{2} d\xi$ and integrated

$$p = c_1 e^{-\xi^2/4}\ . \tag{2.58}$$

Then, from $F' = p$, $u = F$, the similarity solution is

$$u(x,t) = c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\xi^2/4} d\xi + c_2\ . \tag{2.59}$$

The mathematical solution obtained above and other solutions obtained by symmetries attain physical interpretations if appropriate initial/boundary conditions are imposed on the differential equations. Boundary value problems will be treated in Section 2.5 of this chapter for special transformations and as a separate chapter for general transformations in Chapter 9.

2.3. SPIRAL GROUP SYMMETRIES

Spiral groups are mixed combinations of translations and scalings. Two problems will be considered.

Problem 2.5. Consider the ordinary differential equation

$$yy'' - y'^2 = 0 \, , \tag{2.60}$$

where $y = y(x)$. The spiral symmetries can be written as

$$x^* = x + \epsilon a \, , \tag{2.61}$$

$$y^* = e^{\epsilon b} y \, , \tag{2.62}$$

The equation transforms into

$$e^{-2\epsilon b} y^* y^{*\prime\prime} - e^{-2\epsilon b} y^{*\prime 2} = 0 \, . \tag{2.63}$$

Multiplying the equation with $e^{2\epsilon b}$, the transformed equation looks exactly the same equation as the original equation

$$y^* y^{*\prime\prime} - y^{*\prime 2} = 0 \, , \tag{2.64}$$

with no further restrictions on the parameters a and b. One can say that the spiral group transformation (2.61) and (2.62) is admitted by the given equation (2.60). The characteristic equation is

$$\frac{dx}{a} = \frac{dy}{by} \, , \tag{2.65}$$

with a solution

$$y = c_1 e^{mx} \, , \tag{2.66}$$

where $m = b/a$. The above solution already satisfies the original equation and is therefore a group invariant solution for arbitrary m values.

Problem 2.6. Consider the nonlinear PDE

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2}\,, \tag{2.67}$$

where $u = u(x,t)$. The spiral group may be expressed as

$$x^* = x + \epsilon a\,, \tag{2.68}$$
$$t^* = e^{\epsilon b} t\,, \tag{2.69}$$
$$u^* = e^{\epsilon c}\, u\,, \tag{2.70}$$

with ϵ being the group parameter and a, b and c some constants. Alternatively, there might be a translation in the time coordinate and scaling in the spatial coordinate as another spiral group transformation. The equation is transformed into

$$e^{\epsilon(b-c)}\frac{\partial u^*}{\partial t^*} = e^{-2\epsilon c}u^*\frac{\partial^2 u^*}{\partial x^{*2}}\,. \tag{2.71}$$

Multiply the equation by $e^{\epsilon(c-b)}$

$$\frac{\partial u^*}{\partial t^*} = e^{\epsilon(-b-c)}u^*\frac{\partial^2 u^*}{\partial x^{*2}}\,. \tag{2.72}$$

Comparing the transformed equation with the original equation, the invariance conditions are

$$c = -b\,, \quad a \text{ arbitrary.} \tag{2.73}$$

The associated characteristic equation is

$$\frac{dx}{a} = \frac{dt}{bt} = \frac{du}{-bu}\,. \tag{2.74}$$

Integrating the first two equations

$$x = m\ln t + ln\xi\,, \tag{2.75}$$

where ξ is a constant of integration and $m = a/b$. Solving for ξ

$$\xi = \frac{e^x}{t^m}\,. \tag{2.76}$$

Next, the second and third equation is taken and integrated

$$\ln u = -\ln t + \ln F(\xi)\,, \tag{2.77}$$

which yields

$$u = \frac{F(\xi)}{t} \ . \tag{2.78}$$

Equations (2.76) and (2.78) are the similarity variable and function which will transform the PDE into an ODE. First the derivatives should be calculated

$$\frac{\partial u}{\partial t} = -\frac{F}{t^2} - m\frac{e^x}{t^{m+2}}F' \ , \tag{2.79}$$

$$\frac{\partial u}{\partial x} = \frac{e^x}{t^{m+1}}F' \ , \tag{2.80}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{e^x}{t^{m+1}}F' + \frac{e^{2x}}{t^{2m+1}}F'' \ . \tag{2.81}$$

Substituting into the original equation, multiplying by t^2 and rearranging, the transformed ODE is

$$\xi^2 FF'' + \xi FF' + m\xi F' + F = 0 \ , \tag{2.82}$$

which is highly nonlinear. If the equation does not possess an analytical solution, then searching for numerical solutions will be inevitable. However, integrating a transformed ODE is always easier than integrating the original PDE. Once $F(\xi)$ is determined, then the solution is expressed by the aid of (2.78) and (2.76) in terms of the original variables.

2.4. COMBINED SYMMETRIES

All three symmetries previously discussed, namely the translational, scaling and spiral can be combined in a single transformation. An example problem will be solved.

Problem 2.7. Consider the nonlinear heat transfer equation (Pakdemirli and Yürüsoy, 1998)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(e^{mu}\frac{\partial u}{\partial x}\right) \ , \tag{2.83}$$

where $u = u(x,t)$. The combined transformation may be expressed as

$$x^* = e^{\epsilon a}x + \epsilon b \ , \tag{2.84}$$
$$t^* = e^{\epsilon c}t + \epsilon d \ , \tag{2.85}$$
$$u^* = e^{\epsilon e}\, u + \epsilon f \ , \tag{2.86}$$

Translational symmetry corresponds to $a = c = e = 0$, scaling symmetry corresponds to $b = d = f = 0$ and spiral symmetry corresponds to $a = d = f = 0$. Substituting the transformations into the original equation and dividing by the coefficient of the leading term

$$\frac{\partial u^*}{\partial t^*} = e^{\epsilon(2a-c-mf)} \frac{\partial}{\partial x^*}\left(e^{me^{-\epsilon e}u^*} \frac{\partial u^*}{\partial x^*}\right) , \tag{2.87}$$

one can conclude that

$$e = 0 \,, \quad 2a - c - mf = 0 \,, \tag{2.88}$$

for invariance. Parameter f can be solved in terms of the other parameters leaving a four parameter group of transformation

$$x^* = e^{\epsilon a}x + \epsilon b \,, \tag{2.89}$$
$$t^* = e^{\epsilon c}t + \epsilon d \,, \tag{2.90}$$
$$u^* = u + \epsilon\left(\frac{2a-c}{m}\right) . \tag{2.91}$$

In fact, the above transformation is the full group for this equation (Bluman and Kumei, 1989). One of the similarity solutions will be derived by the special choice of $a = c = m$ and $b = d = 0$ yielding the characteristic equation

$$\frac{dx}{mx} = \frac{dt}{mt} = \frac{du}{1} . \tag{2.92}$$

The similarity variable and function

$$\mu = \frac{x}{t} \,, \quad u = \frac{1}{m}\ln t + r(\mu) \,, \tag{2.93}$$

when substituted into the original equation

$$m(e^{mr}r')' + m\mu r' = 1 \,, \tag{2.94}$$

reduces the PDE into a nonlinear ODE.

Before attacking any equation to find the full group which is algebraically involved, the special transformations or the combined one might yield preliminary results with much less effort. Such calculations may be used as a verification of the results of hand calculation or symbolic computer calculations for more involved cases.

2.5. BOUNDARY VALUE PROBLEMS

In all the above equations, no boundary or initial conditions are considered. However, physical problems usually accompany boundary/initial conditions associated with the differential equation. In such cases, many of the mathematical solutions obtained by the group analysis with disregarding the conditions are inappropriate for expressing the physical solutions. This is because the transformations have to be admitted both by the equation and by the conditions which restricts the number of solutions. Chapter 9 is devoted to boundary value problems and their treatment using general transformations. The boundary layer problem of a viscous fluid will be treated next.

Problem 2.8. Consider the dimensionless boundary layer equations with the conditions (Pakdemirli and Yürüsoy, 1998)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\ , \tag{2.95}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+U\frac{dU}{dx}\ , \tag{2.96}$$

$$u(x,0)=0,\ \ v(x,0)=0,\ \ u(x,\infty)=U(x)\ . \tag{2.97}$$

originally derived by Prandtl (1905) with approximation of the Navier-Stokes equations near the boundary. $u(x,y)$ is the x-component of velocity, $v(x,y)$ is the y-component of velocity inside the boundary layer. $U(x)$ is the inviscid velocity outside the boundary layer. By ad hoc techniques, Prandtl proposed a similarity transformation to solve the equations. In fact, as with most of the fluid flow problems, the boundary layer equations usually admit scaling symmetries and a more systematic reduction will be presented other than the one presented in fluid mechanics textbooks. The transformations are

$$x^*=e^{\epsilon a}x\ , \tag{2.98}$$
$$y^*=e^{\epsilon b}y\ , \tag{2.99}$$
$$u^*=e^{\epsilon c}\,u\ , \tag{2.100}$$
$$v^*=e^{\epsilon d}\,v\ , \tag{2.101}$$
$$U^*=e^{\epsilon e}\,U\ , \tag{2.102}$$

Substituting the transformations, dividing all the equations with the leading term coefficients,

$$\frac{\partial u^*}{\partial x^*} + e^{\epsilon(b+c-a-d)}\frac{\partial v^*}{\partial y^*} = 0\,, \tag{2.103}$$

$$u^*\frac{\partial u^*}{\partial x^*} + e^{\epsilon(b+c-a-d)}v^*\frac{\partial u^*}{\partial y^*} = e^{\epsilon(2b+c-a)}\frac{\partial^2 u^*}{\partial y^{*2}} + e^{\epsilon 2(c-e)}U^*\frac{dU^*}{dx^*}\,, \tag{2.104}$$

$$u^*(x^*,0) = 0,\ \ v^*(x^*,0) = 0,\ \ u^*(x^*,\infty) = e^{\epsilon(c-e)}U^*(x^*)\,, \tag{2.105}$$

the invariance conditions turn out to be

$$b + c - a - d = 0\,,\ \ 2b + c - a = 0\,,\ \ c - e = 0\ . \tag{2.106}$$

Expressing all parameters in terms of a and c

$$b = \frac{a-c}{2}\ ,\quad d = \frac{c-a}{2}\ ,\quad e = c\,, \tag{2.107}$$

the scaling transformation admitted by the equation is

$$x^* = e^{\epsilon a}x\,, \tag{2.108}$$
$$y^* = e^{\epsilon(a-c)/2}y\,, \tag{2.109}$$
$$u^* = e^{\epsilon c}\,u\,, \tag{2.110}$$
$$v^* = e^{\epsilon(c-a)/2}\,v\,, \tag{2.111}$$
$$U^* = e^{\epsilon c}U\,. \tag{2.112}$$

Assuming $c = ma$ without loss of generality, the associate equations with the group are

$$\frac{dx}{x} = \frac{dy}{(\frac{1-m}{2})y} = \frac{du}{mu} = \frac{dv}{(\frac{m-1}{2})v} = \frac{dU}{mU}\ . \tag{2.113}$$

Solving the differential equation system, the similarity variable and functions are

$$\xi = yx^{(m-1)/2}\,,\ \ u = x^m f(\xi)\,,\quad v = x^{(m-1)/2}g(\xi)\,,\ \ u = kx^m, \tag{2.114}$$

where k is a constant. The similarity variable and functions transform the PDE system into an ODE system

$$mf + \frac{m-1}{2}\xi f' + g' = 0\,, \tag{2.115}$$

$$mf^2 + \frac{m-1}{2}\xi ff' + gf' = f'' + mk^2, \tag{2.116}$$

$$f(0) = 0\,,\ \ g(0) = 0\,,\ f(\infty) = k\ . \tag{2.117}$$

Note that, if the special symmetry is a symmetry of the differential equation but not of the conditions, the group transformation cannot transform appropriately the conditions and hence, cannot be used as a solution of the boundary value problem. Since the group is applied to the conditions also, the transformation above is an admissible transformation.

The ODE system can be solved numerically which is definitely an easier task than solving the original PDE. Then (2.114) will yield the solution in terms of the original variables.

2.6. GENERAL FORMS OF DIFFERENTIAL EQUATIONS ADMITTING SPECIAL GROUPS

In this section, the reverse problem is considered, that is, the most general form of a differential equation admitted by a given special symmetry will be derived. With the aid of the theorems given, the special symmetry can be identified by a preliminary inspection of the equation.

Theorem 2.1. The nonlinear ordinary differential equation

$$F\left(y', y'', \dots, y^{(k)}\right) = 0 , \tag{2.118}$$

with $y^{(i)} = \frac{d^i y}{dx^i}$, $y^{(0)} = y$, $i = 0,1,2, \dots k$ admits the translational symmetries

$$x^* = x + \epsilon a , \tag{2.119}$$

$$y^* = y + \epsilon b \tag{2.120}$$

Proof

Since the derivatives are transformed as

$$y^{*(i)} = y^{(i)},\ i = 1,2, \dots k , \tag{2.121}$$

the equation takes the form

$$F\left(y^{*\prime}, y^{*\prime\prime}, \dots, y^{*(k)}\right) = 0 , \tag{2.122}$$

which indicates that the equation is invariant under the given transformation. In fact, if a variable is missing in the equations, then translational symmetry of the missing variable is admitted by the equation.

As an example

$$y''' + y'y''^2 = 0\ , \tag{2.123}$$

admits translational symmetries in both *x* and *y* directions.

Theorem 2.2. The nonlinear ordinary differential equation

$$F\left(y, y', y'', \dots, y^{(k)}\right) = 0 \tag{2.124}$$

with $y^{(i)} = \frac{d^i y}{dx^i}$, $y^{(0)} = y$, $i = 0,1,2,\dots k$ admits the translational symmetry

$$x^* = x + \epsilon a\ , \tag{2.125}$$

$$y^* = y \tag{2.126}$$

A similar proof follows for the above theorem. Since y is not missing, the translations in the *y* coordinate is not allowed.

As an example

$$y'' + \sin(y')y = 0\ , \tag{2.127}$$

admits translations in the *x* direction only.

Theorem 2.3. If the nonlinear ordinary differential equation

$$F\left(x, y, y', y'', \dots, y^{(k)}\right) = 0\ , \tag{2.128}$$

accepts the scaling transformations

$$x^* = e^{\epsilon a}x\ , \tag{2.129}$$

$$y^* = e^{\epsilon b}y\ , \tag{2.130}$$

then the equation can be cast into the following form

$$F\left(x^{-b/a}y, x^{1-b/a}y', x^{2-b/a}y'', \dots, x^{k-b/a}y^{(k)}\right) = 0 \tag{2.131}$$

Proof

Substitute the scaling transformations to the original equation (2.128),

$$F\left(e^{-\epsilon a}x^*, e^{-\epsilon b}y^*, e^{\epsilon(a-b)}y^{*\prime}, \dots, e^{\epsilon(ka-b)}y^{*(k)}\right) = 0\ . \tag{2.132}$$

It is clear that for an arbitrary F function, the invariance requires $a = b = 0$, hence the most general form does not accept scaling symmetries. However, for some special forms of F, the scaling symmetry may be admitted by the equation. To find this special form, differentiate F with respect to the group parameter ϵ and evaluate the derivatives at $\epsilon = 0$

$$\left[F_x \frac{dx}{d\epsilon} + F_y \frac{dy}{d\epsilon} + F_{y'} \frac{dy'}{d\epsilon} + \cdots + F_{y^{(k)}} \frac{dy^{(k)}}{d\epsilon}\right]_{\epsilon=0} = 0 , \qquad (2.133)$$

or since $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = -ax$, $\left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = -by$, $\left.\frac{dy'}{d\epsilon}\right|_{\epsilon=0} = -(b-a)y'$,

$\left.\frac{dy^{(k)}}{d\epsilon}\right|_{\epsilon=0} = -(b-ka)y^{(k)}$, the equation takes the form

$$axF_x + byF_y + (b-a)y'F_{y'} + \cdots + (b-ka)y^{(k)}F_{y^{(k)}} = 0 . \qquad (2.134)$$

The above equation can be solved by the method of characteristics

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{dy'}{(b-a)y'} = \cdots = \frac{dy^{(k)}}{(b-ka)y^{(k)}} = \frac{dF}{0} , \qquad (2.135)$$

with the solutions substituted back into the original equation

$$F\left(x^{-b/a}y, x^{1-b/a}y', x^{2-b/a}y'', \ldots, x^{k-b/a}y^{(k)}\right) = 0 . \qquad (2.136)$$

For uniform scaling symmetry, $b = a$ and the above form reduces to

$$F\left(x^{-1}y, y', xy'', \ldots, x^{k-1}y^{(k)}\right) = 0 . \qquad (2.137)$$

Problem 2.9. The ordinary differential equation

$$y'' + \frac{y'^3}{x} = 0 , \qquad (2.138)$$

admits uniform scaling since the equation can be written as

$$xy'' + y'^3 = 0 , \qquad (2.139)$$

which is a special form of $F(y', xy'') = 0$.

Problem 2.10. Determine whether the equation

$$y'' + xy'^2 = 0 , \qquad (2.140)$$

admits scaling symmetry and the form of the symmetry.

Solution

Multiply the equation by x^m to cast into the standard form.

$$x^m y'' + x^{m+1} y'^2 = 0 \; . \tag{2.141}$$

Compare the equation with (2.136)

$$m = 2 - \frac{b}{a} \; , \quad m + 1 = 2\left(1 - \frac{b}{a}\right) . \tag{2.142}$$

Eliminating b/a between the equations, $m = 3$. Then $b = -a$ and the special scaling group is

$$x^* = e^{\epsilon a} x \tag{2.143}$$
$$y^* = e^{-\epsilon a} y \tag{2.144}$$

where parameter a may be augmented into the group parameter ϵ without loss of generality.

Problem 2.11. Show that the differential equation

$$y y''' + y'' = 0 \; , \tag{2.145}$$

admits uniform scaling symmetry.

Solution

Multiply the equation by x

$$x y y''' + x y'' = 0 \; , \tag{2.146}$$

and express the first term as

$$x^{-1} y \, x^2 y''' + x y'' = 0 \; , \tag{2.147}$$

which is definitely in the form of $F(x^{-1}y, xy'', x^2 y''') = 0$ and accepts uniform scaling. Such tests of course can be done alternatively by direct application of the symmetries to the differential equations also.

Theorem 2.4. If the nonlinear ordinary differential equation

$$F\left(x, y, y', y'', \ldots, y^{(k)}\right) = 0 \; , \tag{2.148}$$

accepts translational symmetries

$$x^* = x + \epsilon a \ , \tag{2.149}$$
$$y^* = y + \epsilon b \ , \tag{2.150}$$

then the equation can be cast into the following form

$$F\left(y - \frac{b}{a}x, y', y'', \dots, y^{(k)}\right) = 0 \ . \tag{2.151}$$

For the special case of $b = a$, the form is

$$F\left(y - x, y', y'', \dots, y^{(k)}\right) = 0 \tag{2.152}$$

The proof is left as an exercise.

Problem 2.12. The equation $F\left(y', y'', \dots, y^{(k)}\right) = 0$ definitely accepts translational symmetries (2.149) and (2.150) for arbitrary a and b parameters since it is a special form of (2.151).

Problem 2.13. The equation $y''' + (y - 2x)y' = 0$ accepts the special translational symmetry for $b = 2a$ which makes the equation a special form of (2.151).

Theorem 2.5. If the nonlinear ordinary differential equation

$$F\left(x, y, y', y'', \dots, y^{(k)}\right) = 0 \ , \tag{2.153}$$

accepts a combination of scaling, spiral and translational symmetries

$$x^* = e^{\epsilon a}x + \epsilon b \ , \tag{2.154}$$
$$y^* = e^{\epsilon c}y + \epsilon d \ , \tag{2.155}$$

then the equation can be cast into the following form

$$F\left((cy + d)(ax + b)^{-c/a}, (ax + b)^{1-c/a}y', \dots, (ax + b)^{k-c/a}y^{(k)}\right) = 0 \tag{2.156}$$

Proof

Substitute the transformations (2.154) and (2.155) into the original equation (2.153)

$$F\left(e^{-\epsilon a}(x^* - \epsilon b), e^{-\epsilon c}(y^* - \epsilon d), e^{\epsilon(a-c)}y^{*\prime}, \dots, e^{\epsilon(ka-c)}y^{*(k)}\right) = 0. \tag{2.157}$$

It is clear that for an arbitrary F function, the invariance requires $a = b = c = d = 0$, hence the most general form does not accept combined symmetries. However, for some special forms of F, the combined symmetries may be admitted by the equation. To find this special form, differentiate F with respect to the group parameter ϵ and evaluate the derivatives at $\epsilon = 0$

$$\left[F_x \frac{dx}{d\epsilon} + F_y \frac{dy}{d\epsilon} + F_{y'} \frac{dy'}{d\epsilon} + \cdots + F_{y^{(k)}} \frac{dy^{(k)}}{d\epsilon}\right]_{\epsilon=0} = 0 \,, \tag{2.158}$$

or since $\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0} = -(ax+b)$, $\left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = -(cy+d)$, $\left.\frac{dy'}{d\epsilon}\right|_{\epsilon=0} = -(c-a)y'$,

$\left.\frac{dy^{(k)}}{d\epsilon}\right|_{\epsilon=0} = -(c-ka)y^{(k)}$, the equation takes the form

$$(ax+b)F_x + (cy+d)F_y + (c-a)y'F_{y'} + \cdots + (c-ka)y^{(k)}F_{y^{(k)}} = 0 \,. \tag{2.159}$$

The above equation can be solved by the method of characteristics

$$\frac{dx}{ax+b} = \frac{dy}{cy+d} = \frac{dy'}{(c-a)y'} = \cdots = \frac{dy^{(k)}}{(c-ka)y^{(k)}} = \frac{dF}{0} \,, \tag{2.160}$$

with the solutions substituted back into the original equation

$$F\left((cy+d)(ax+b)^{-c/a}, (ax+b)^{1-c/a}y', \ldots, (ax+b)^{k-c/a}y^{(k)}\right) = 0 \,. \tag{2.161}$$

Note that $a \neq 0$ and $c \neq 0$ to prevent any singularity. Actually $a = c = 0$ corresponds to the translational symmetries the general form of which was already given in Theorem 2.4

Problem 2.14. Show that the equation

$$(x-2)^2 y'' + (x-2)y' + 3y = 0 \,, \tag{2.162}$$

accepts a special form of the combined symmetries given in (2.154) and (2.155).

Solution

Divide the equation by $(x-2)^3$

$$\frac{y''}{(x-2)} + \frac{y'}{(x-2)^2} + \frac{3y}{(x-2)^3} = 0 \ . \tag{2.163}$$

Comparing (2.163) with the general form (2.161), one may conclude that $a = 1, b = -2, c = 3, d = 0$. Hence the specific transformation admitted by the equation is

$$x^* = e^{\epsilon} x - 2\epsilon \ , \tag{2.164}$$
$$y^* = e^{3\epsilon} y \ , \tag{2.165}$$

which can be verified by direct substitution into the original equation.

Theorem 2.6. If the second order nonlinear partial differential equation with two independent variables

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}\right) = 0 \ , \tag{2.166}$$

accepts non-uniform scaling symmetry

$$x^* = e^{\epsilon a} x \ , \tag{2.167}$$
$$y^* = e^{\epsilon b} y \ , \tag{2.168}$$
$$u^* = e^{\epsilon c} u \ , \tag{2.169}$$

then the equation can be cast into the following form

$$F\left(\frac{y}{x^{b/a}}, \frac{u}{x^{c/a}}, \frac{u_x}{x^{(c-a)/a}}, \frac{u_y}{x^{(c-b)/a}}, \frac{u_{xx}}{x^{(c-2a)/a}}, \frac{u_{yy}}{x^{(c-2b)/a}}, \frac{u_{xy}}{x^{(c-a-b)/a}}\right) = 0 \tag{2.170}$$

Proof follows similar steps with the previous theorems.

Problem 2.15. Show that the heat equation admits a scaling symmetry and find the form of that symmetry

$$u_t = u_{xx} \ , \tag{2.171}$$

Solution

Change the notation $t = x$, $x = y$. Comparing (2.171) with (2.170), to look similar, powers of the variable x should be the same so that they can be cancelled

$$\frac{u_x}{x^{(c-a)/a}} = \frac{u_{yy}}{x^{(c-2b)/a}} \ , \tag{2.172}$$

which requires $c - a = c - 2b$ or $a = 2b$ and c remains arbitrary. Hence the specific transformation is

$$x^* = e^{\epsilon 2b} x \ , \tag{2.173}$$
$$y^* = e^{\epsilon b} y \ , \tag{2.174}$$
$$u^* = e^{\epsilon c} u \ . \tag{2.175}$$

By direct substitution into the original equation, the same transformations were already found in Problem 2.4.

2.7. GROUP CLASSIFICATION FOR SPECIAL TRANSFORMATIONS

In the previous chapter, the general form of the differential equation is determined for which a special transformation is admitted. In this section, the equation may contain arbitrary functions or parameters which are not the dependent variables themselves and for some special forms of these functions, the symmetries may be richer. This problem of determining all forms which increase the symmetries is a group classification problem. It will be investigated in detail later in Chapter 8 for the general Lie group of transformations. Here, only the group classification with respect to special transformations will be outlined in a sample problem.

Problem 2.16. Consider the nonlinear dimensionless fin equation (Aziz and Na, 1981; Pakdemirli and Şahin, 2006a, 2006b)

$$\frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - N^2 f(x)\theta = \frac{\partial \theta}{\partial t} \ , \tag{2.176}$$

where x is the spatial variable and t is the time. $\theta(x,t)$ is the dimensionless temperature. The thermal conductivity $k(\theta)$ is an arbitrary function of temperature and the heat transfer coefficient $f(x)$ is an arbitrary function of the spatial variable with *N* being the fin parameter. Scaling group classification will be performed for the thermal conductivity and heat transfer coefficient

$$x^* = \lambda^a x \ , \tag{2.177}$$
$$t^* = \lambda^b t \ , \tag{2.178}$$
$$\theta^* = \lambda^c \theta \ , \tag{2.179}$$

Inserting the transformations and equating the coefficient of the right hand side term to 1, the transformed equation is

$$\lambda^{2a-b}\frac{\partial}{\partial x^*}\left(k(\lambda^{-c}\theta^*)\frac{\partial\theta^*}{\partial x^*}\right)-N^2\lambda^{-b}f(\lambda^{-a}x^*)\theta^*=\frac{\partial\theta^*}{\partial t^*}\,. \qquad (2.180)$$

Comparing (2.180) with (2.176), the invariance conditions are

$$\lambda^{2a-b}k(\lambda^{-c}\theta^*)=k(\theta)\,, \qquad (2.181)$$

$$\lambda^{-b}f(\lambda^{-a}x^*)=f(x)\,. \qquad (2.182)$$

If $k(\theta)$ and $f(x)$ are arbitrary, then the only choice from above is $a=b=c=0$ which shows that the equation does not accept scaling symmetry for the most general case. However, for some specific forms of the functions, the equation may admit such symmetries and the goal is to find them. Differentiating the above equations with respect to λ and returning back to the original variables gives the equations for classifications

$$(2a-b)k(\theta)-c\theta k'(\theta)=0\,, \qquad (2.183)$$

$$bf(x)+axf'(x)=0\,. \qquad (2.184)$$

The first equation, i.e. Eq. (2.183) includes three distinct cases

i) $k=k(\theta)$

Since the thermal conductivity is arbitrary, the coefficients in (2.183) should vanish leading to $b=2a$ and $c=0$. If further the heat transfer coefficient is also arbitrary, then $a=b=0$ from (2.184) which annihilates the symmetries. However, for $b=2a$, equation (2.184) has a solution $f=1/x^2$. Hence, the special transformation for this case is

$$x^*=\lambda^a x\,,\ t^*=\lambda^{2a}t\,,\ \theta^*=\theta\,. \qquad (k=k(\theta), f=1/x^2) \quad (2.185)$$

ii) $k=k_0$

Since the thermal conductivity is a constant, from (2.183) parameter c remains arbitrary and $b=2a$. If further $f(x)$ is arbitrary, then $a=b=0$. The transformation for this case is

$$x^*=x\,,\ t^*=t\,,\ \theta^*=\lambda^c\theta\,. \quad (k=k_0, f=f(x)) \qquad (2.186)$$

If one does not require $f(x)$ to be arbitrary, then solving (2.184) for $b=2a$ yields $f=1/x^2$. Hence the scaling symmetries increase for this case

$$x^* = \lambda^a x\ ,\ t^* = \lambda^{2a} t\ ,\ \theta^* = \lambda^c \theta\ . \quad (k = k_0, f = 1/x^2) \tag{2.187}$$

iii) $k = k_0\theta^\beta$

Integrating (2.183) leads directly to this case with $\beta = (2a - b)/c$. There sub-cases for heat transfer coefficient exist: If $f(x)$ is arbitrary, then $a = b = 0$ which implies $\beta = 0$ which is the previous constant conduction case. For $f = 1$, $b = 0$ and the symmetries are

$$x^* = \lambda^a x\ ,\ t^* = t\ ,\ \theta^* = \lambda^{2a/\beta} \theta\ . \quad (k = k_0\theta^\beta, f = 1) \tag{2.188}$$

The richest symmetries occur if (2.184) is integrated leading to $f(x) = x^m$ where $m = -b/a$. The transformations are

$$x^* = \lambda^a x\ ,\ t^* = \lambda^{-ma} t\ ,\ \theta^* = \lambda^{(2+m)a/\beta} \theta \quad (k = k_0\theta^\beta, f = x^m) \tag{2.189}$$

Results are summarized in Table 2.1.

Table 2.1. Scaling symmetry group classification for the fin equation

Conductivity	Heat Transfer Coefficient	Scaling Symmetries
$k(\theta)$	$f(x)$	$x^* = x\ ,\ t^* = t\ ,\ \theta^* = \theta$
	$1/x^2$	$x^* = \lambda^a x\ ,\ t^* = \lambda^{2a} t\ ,\ \theta^* = \theta$
k_0	$f(x)$	$x^* = x\ ,\ t^* = t\ ,\ \theta^* = \lambda^c \theta$
	$1/x^2$	$x^* = \lambda^a x\ ,\ t^* = \lambda^{2a} t\ ,\ \theta^* = \lambda^c \theta$
$k_0\theta^\beta$	$f(x)$	$\beta = 0$, reduces to the case of k_0
	1	$x^* = \lambda^a x\ ,\ t^* = t\ ,\ \theta^* = \lambda^{2a/\beta} \theta$
	x^m	$x^* = \lambda^a x,\ t^* = \lambda^{-ma} t\ ,\ \theta^* = \lambda^{(2+m)a/\beta} \theta$

Group classification with respect to spiral and translational symmetries are left as exercises.

2.8. EXERCISES

E2.1. Show that the ordinary differential equation

$$y'' + y = 2x\ ,$$

admits the translational symmetry $x^* = x + \epsilon a,\ y^* = y + \epsilon b$ for the condition of $b = 2a$, and show that the group invariant solution is

$$y = 2x\ .$$

E2.2. Show that the ordinary differential equation

$$y''' + y'y'' = 0\ ,$$

admits the translational symmetry $x^* = x + \epsilon a$, $y^* = y + \epsilon b$ for arbitrary a and b parameters, and show that the group invariant solution is

$$y = mx + c_1,$$

where $m = b/a$.

E2.3. Show that the dimensionless heat conduction equation

$$u_t = u_{xx}\ ,$$

admits the translational symmetries $x^* = x + \epsilon a$, $t^* = t + \epsilon b$, $u^* = u + \epsilon c$ for arbitrary a, b and c parameters, the corresponding similarity variables are

$$\mu = x - mt\ ,\quad u = nx + f(\mu)\ ,$$

where $m = a/b$ and $n = c/a$. Show that the equation reduces to the following ODE

$$f'' + mf' = 0\ ,$$

and solving the above equation, show that one group invariant solution is

$$u(x,t) = nx + c_1 + c_2 e^{-m(x-mt)}.$$

E2.4. Show that the Cauchy-Euler differential equation

$$x^2 y'' + xy' - 9y = 0$$

admits the scaling symmetry $x^* = e^{\epsilon a}x$, $y^* = e^{\epsilon b}y$ for arbitrary a and b parameters, and show that the group invariant solution is

$$y = c_1 x^3 + c_2 x^{-3},$$

which is the general solution of the equation.

E2.5. Show that the nonlinear ordinary differential equation

$$y'' - yy' = 0\ ,$$

admits the scaling symmetry $x^* = e^{\epsilon a}x$, $y^* = e^{\epsilon b}y$ for $b = -a$, and show that the group invariant solution corresponding to this symmetry is

$$y = -\frac{2}{x} \, .$$

E2.6. Show that the dimensionless wave equation

$$u_{tt} = u_{xx} \, ,$$

admits the scaling symmetries $x^* = \lambda^a x$, $t^* = \lambda^b t$, $u^* = \lambda^c u$ for $b = a$ and c arbitrary. Using the transformation, show that the corresponding similarity variables are

$$\xi = \frac{x}{t} \, , \quad u = x^m f(\xi) \, ,$$

where $m = c/a$. Show that the equation reduces to the following ODE under the similarity transformation

$$(1 - \xi^2) f'' + 2\left(\frac{m}{\xi} - \xi\right) f' + \frac{m(m-1)}{\xi^2} f = 0 \, .$$

E2.7. Show that the nonlinear ordinary differential equation

$$yy' + 3y^2 = 0 \, ,$$

admits the spiral symmetry $x^* = x + \epsilon a$, $y^* = e^{\epsilon b}y$ for arbitrary a and b parameters and show that the group invariant solution corresponding to this symmetry is

$$y = c_1 e^{-3x} \, .$$

E2.8. Show that the nonlinear ordinary differential equation

$$y'y'' - 8y^2 = 0 \, ,$$

admits the spiral symmetry $x^* = x + \epsilon a$, $y^* = e^{\epsilon b}y$ for arbitrary a and b parameters and show that one of the group invariant solutions corresponding to this symmetry is

$$y = c_1 e^{2x} \, .$$

E2.9. Show that the first order nonlinear partial differential equation

$$u_t = (u_{xx})^2,$$

admits the spiral symmetries $x^* = x + \epsilon a,\ t^* = e^{\epsilon b}t,\ u^* = e^{\epsilon c}u$ for $c = -b$ and a arbitrary. Using the transformation, show that the corresponding similarity variables are

$$\xi = \frac{e^x}{t^m}, \quad u = \frac{f(\xi)}{t},$$

where $m = a/b$. Show that the equation reduces to the following ODE under the similarity transformation

$$\xi^2 f'^2 + m\xi f' + f = 0 .$$

E2.10. Show that the nonlinear heat transfer equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right),$$

admits the combined symmetries

$$x^* = e^{\epsilon a}x + \epsilon b$$
$$t^* = e^{\epsilon c}t + \epsilon d$$
$$u^* = e^{\epsilon e}\,u + \epsilon f$$

for the conditions of $f = 0, e = 2a - c$ with a, b, c and d arbitrary. Choosing $b = 0$, $c = 2a$, show that the similarity variables are

$$\xi = \frac{x}{\sqrt{2at+d}}, \quad u = f(\xi),$$

which reduces the PDE into an ODE

$$(ff')' + a\xi f' = 0 .$$

E2.11. Reconsider the dimensionless boundary layer equations with the conditions (Pakdemirli and Yürüsoy, 1998)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + U\frac{dU}{dx},$$

$$u(x,0) = 0,\ v(x,0) = 0,\ u(x,\infty) = U(x) .$$

For the spiral transformations

$$x^* = x + \epsilon a \, ,$$
$$y^* = e^{\epsilon b} y \, ,$$
$$u^* = e^{\epsilon c} \, u \, ,$$
$$v^* = e^{\epsilon d} \, v \, ,$$
$$U^* = e^{\epsilon e} \, U \, ,$$

show that invariance requires

$$b + c - d = 0, \;\; 2b + c = 0, \;\; c - e = 0, \;\; a \text{ arbitrary.}$$

Assuming $c = ma$, express all parameters in terms of the parameter a. Show that the similarity variable and functions for this choice are

$$\zeta = y e^{(m/2)x} \; , \quad u = e^{mx} p(\zeta) \quad , \quad v = e^{(m/2)x} q(\zeta) \quad , \quad U = k e^{mx} \, ,$$

where k is a constant. Show that the similarity variable and functions transform the PDE system into an ODE system

$$mp + \frac{m}{2} \zeta p' + q' = 0 \, ,$$

$$mp^2 + \frac{m}{2} \zeta p p' + q p' = p'' + m k^2 \, ,$$

$$p(0) = 0 \, , \;\; q(0) = 0 \, , \;\; p(\infty) = k \; .$$

E2.12. Prove that for the nonlinear ordinary differential equation

$$F\left(x, y', y'', \ldots, y^{(k)}\right) = 0 \, ,$$

the translational symmetry admitted is $x^* = x$, $y^* = y + \epsilon a$.

E2.13. For the given equations, determine the type of translational symmetries admitted by the equations (Hint: Use Theorems 2.1, 2.2 and Exercise 2.12 for faster results)

a) $y'' + \cos(y') = 0$,

b) $x^2 y''' - 3y' = 0$,

c) $(1 + y'^2) y'' + y' = 0$.

E2.14. Using Theorem 2.3, show that the equation

$$yy'' + \ln(y') = 0 \, ,$$

admits uniform scaling.

E2.15. Using Theorem 2.3, show that the equation

$$x^4 y'' - \sin(x^3 y') + \cos(x^2 y) = 0 \, ,$$

admits non-uniform scaling and determine the specific type of the transformation.

E2.16. Prove Theorem 2.4.

E2.17. Using Theorem 2.4, show that the equation

$$(x + y)y'' + y^2 + 2xy + x^2 = 0 \, ,$$

admits translational symmetries and determine the specific type of the transformation.

E2.18. Using Theorem 2.5, show that the equation

$$(x + 1)^2 y'' + (x + 1)y' - y + 1 = 0 \, ,$$

admits the combined transformations and determine the specific type of the transformation.

E2.19. Using Theorem 2.6, show that the dimensionless wave equation

$$u_{tt} = u_{xx} \, ,$$

admits a scaling symmetry and find the form of that symmetry.

E2.20. Using Theorem 2.6, show that the dimensionless partial differential equation

$$u_x^2 + uu_{xy} = 0 \, ,$$

admits a scaling symmetry and find the form of that symmetry.

E2.21. Perform the group classification for the dimensionless fin equation (Pakdemirli and Şahin, 2006b)

$$\frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - N^2 f(x)\theta = \frac{\partial \theta}{\partial t},$$

with respect to translational symmetries.

E2.22. Perform the group classification for the dimensionless fin equation (Pakdemirli and Şahin, 2006b)

$$\frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - N^2 f(x)\theta = \frac{\partial \theta}{\partial t}$$

with respect to spiral group of symmetries.