Mehmet Pakdemirli

Solutions of Differential Equations by Symmetry Methods

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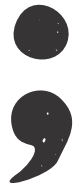

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PREFACE

This book is the outcome of my lecture notes of the graduate courses "Applications of Lie Groups I & II" given in Manisa Celal Bayar University. It also contains substantial material from my published papers on symmetries of differential equations. As a prerequisite, to gain the utmost benefit from the book, a familiarity with the knowledge of ordinary and partial differential equations and their solution techniques would be beneficial. The book can be considered as an introduction to the topic of symmetries of differential equations and how to use them to produce analytical solutions. It is intended to be a graduate textbook on the topic as well as a guide for researchers seeking for analytical solutions to their specific problems. The concentration is on the applications and solution techniques with the theory kept to a minimum so that the book can be readable by engineers and applied oriented researchers without much difficulty. At the end of each chapter, some exercises are also included so that the book can be adapted in a graduate course as a textbook.

As a researcher with an engineering background, I became familiar with the symmetry methods during my Ph.D. studies under the supervision of late Prof. Erdoğan Şuhubi from Istanbul Technical University. His lectures and guidance of symmetry methods in the language of exterior differential forms was my initial familiarity with the topic. Later, I switched to the more classical and direct approach of the Lie group theory instead of calculating the symmetries via exterior differential forms. This book employs the direct classical approach of Lie groups to determine the symmetries of differential equations and finding their solutions. I hope the book will be beneficial to engineers, applied physicists and applied mathematicians who seek solutions to their problems.

Prof. Mehmet Pakdemirli
February 2026, Manisa

CONTENTS

1. Introduction **1**
2. Solutions by Special Group Transformations **4**
2.1. Translational Symmetries 5
2.2. Scaling Symmetries 8
2.3. Spiral Group Symmetries 12
2.4. Combined Symmetries 14
2.5. Boundary Value Problems 16
2.6. General Forms of Differential Equations Admitting Special Groups 18
2.7. Group Classification for Special Transformations 25
2.8. Exercises 27
3. Lie Groups, Infinitesimal Generators and Lie Algebras **34**
3.1. Lie Groups of Transformations 34
3.2. Infinitesimal Transformations and Generators 35
3.3. Invariance of a Curve and Family of Curves 40
3.4. Multi-parameter Lie Group of Transformations 43
3.5. Exercises 47
4. Ordinary Differential Equations: Symmetry Generators **49**
4.1. Extended Groups and Infinitesimal Generators 49
4.2. Invariance of a Differential Equation 53
4.3. First Order Differential Equations 54
4.4. Second Order Differential Equations 57
4.5. Higher Order Differential Equations 62
4.6. Exercises 66
5. Ordinary Differential Equations: Symmetry Solutions **68**
5.1. First Order Differential Equations 68
5.2. Second Order Differential Equations 77
5.3. Higher Order Differential Equations 84
5.4. Determining the General Form from a Given Symmetry 92
5.5. Exercises 96
6. Partial Differential Equations: Symmetry Generators **100**
6.1. Extended Groups and Infinitesimal Generators 100
6.2. Invariance of a Partial Differential Equation 104
6.3. First Order Partial Differential Equations 105

6.4. Second Order Partial Differential Equations 108
6.5. Higher Order Partial Differential Equations 112
6.6. Coupled Partial Differential Equations 115
6.7. Exercises 119
7. Partial Differential Equations: Symmetry Solutions 121
7.1. First Order Partial Differential Equations 122
7.2. Second Order Partial Differential Equations 127
7.3. Higher Order Partial Differential Equations 131
7.4. Coupled Partial Differential Equations 134
7.5. Exercises 136
8. Group Classification Problems 140
8.1. Ordinary Differential Equations 140
8.2. Partial Differential Equations 144
8.3. Exercises 154
9. Boundary Value Problems 155
9.1. Nonlinear PDEs 156
9.2. Finite versus Infinite Domains 159
9.3. Group Classification for Conditions 161
9.4. Exercises 164
10. Approximate Symmetry Theories 167
10.1. Approximate Symmetry Generators 168
10.2. Approximate Solutions 174
10.3. Exercises 179
11. More Recent Approximate Symmetry Theories 181
11.1. Definitions of Approximate Symmetry Theories 181
11.2. Approximate Symmetry Calculations 184
11.3. Exact and Approximate Solutions 186
11.4. Conclusions 187
11.5. Exercises 188
Further Topics 189
References 193