

Chapter 15: Weak Convergence of Sequences of Functions

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Abstract: Weak convergence is a fundamental concept in functional analysis, capturing the limiting behaviour of sequences in normed spaces via duality rather than norm topology. In this paper, we study the space $\mathcal{D}[W, f]$ of weakly convergent sequences of functions in a normed space X . We investigate its algebraic and geometric properties, including linearity, symmetry, solidity, monotonicity, convexity, uniform convexity, reflexivity, and completeness. These results provide a comprehensive framework for understanding weak convergence of function sequences and their applications in functional analysis and operator theory.

Keywords: Weak convergence, solid, monotone, symmetry, convex, uniform convex.

Introduction

Convergence theory forms a cornerstone of functional analysis, providing a systematic framework for studying the limiting behaviour of sequences in normed linear spaces. In finite dimensional settings, norm (or strong) convergence is the most natural and intuitive notion, ensuring that the distance between successive terms and the limit tends to zero. However, in infinite-dimensional spaces, norm convergence can be too restrictive or even fail to exist for certain sequences of interest. This limitation has motivated the study of weaker modes of convergence, which capture the asymptotic behaviour of sequences under duality or functional evaluation. Among these, *weak convergence* is one of the most fundamental concepts, particularly for analyzing the structure of Banach and Hilbert spaces [1,2].

The concept of weak convergence was formally introduced by Stefan Banach in his seminal monograph *Theory of Linear Operations* [4]. A sequence $\{x_n\}$ in a normed space X is said to converge weakly to $x \in X$ if, for every bounded linear functional $\phi \in X^*$ the sequence of scalars $\{\phi(x_n)\}$ converges to $\phi(x)$. This notion highlights a fundamental principle in functional analysis: the study of a normed space is closely intertwined with the properties of its dual space. Notably, while norm convergence implies weak convergence to the same limit, the converse is generally false [5,6]. This distinction underscores the subtleties of infinite dimensional analysis, where sequences can converge under all bounded linear functionals yet fail to converge in norm.

The *Principle of Uniform Boundedness* (Banach–Steinhaus theorem) plays a central role in the development of weak convergence theory [7,17,3]. It ensures that a family of bounded linear functionals that converge pointwise is uniformly bounded, which in turn provides foundational tools for analyzing