T. C. Vijayaraghavan

# The Complexity of Logarithmic Space Bounded Counting Classes

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ABSTRACT. In this monograph, we study complexity classes that are defined using $O(\log n)$-space bounded non-deterministic Turing machines. We prove salient results of Computational Complexity in this topic such as the Immerman-Szelepcsenyi Theorem, the Isolating Lemma, theorems of M. Mahajan and V. Vinay on the determinant and many consequences of these very important results. The manuscript is intended to be a comprehensive textbook on the topic of The Complexity of Logarithmic Space Bounded Counting Classes.

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# Preface

It is a fact that the study of complexity classes has grown enormously and an innumerable number of results have been proved on almost every complexity class that has been defined. A counting class is defined as any complexity class whose definition is based on a function of the number of accepting computation paths and/or the number of rejecting computation paths of a non-deterministic Turing machine. By restricting the space used by a non-deterministic Turing machine to be $O(\log n)$, where $n$ is the size of the input, we can define many logarithmic space bounded counting complexity classes. The first and fundamental logarithmic space bounded counting complexity class that one can easily define is Non-deterministic Logarithmic space ($\mathrm{NL}$). The definition of any other logarithmic space bounded counting complexity class is based on $\mathrm{NL}$.

The purpose of this research monograph is to serve as a textbook for teaching the topic of logarithmic space bounded counting complexity classes and almost all the salient results on them. In this monograph, we introduce the beautiful and sophisticated theory of logarithmic space bounded counting complexity classes. In Chapter 1, we briefly introduce the Turing machine model and the Boolean circuit model of computation. In Chapters 2 to 5 of this monograph, we define logarithmic space bounded counting complexity classes and we prove structural properties of these complexity classes. In particular we study some important results on a number of complexity classes whose definitions are based on Turing machines and which are contained between the circuit complexity classes $\mathrm{NC}^1$ and $\mathrm{NC}^2$. The complexity classes of interest to us are $\mathrm{NL}$, $\sharp\mathrm{L}$, $\mathrm{GapL}$, $\mathrm{C_{=}L}$, $\mathrm{UL}$, $\mathrm{Mod}_p\mathrm{L}$, $\mathrm{Mod}_k\mathrm{L}$, ModL and PL, where $k, p \in \mathbb{N}$, $k \geq 2$ and $p$ is a prime. Results we prove in Chapters 2 to 5 are diverse in the ideas and techniques involved such as the following:

- $\star$ the non-deterministic counting method invented to prove $\mathrm{NL} = $ co-NL which is the Immerman-Szelepcsenyi Theorem in the logarithmic space setting and some of its useful consequences which is to show that $\mathrm{L}^{\mathrm{NL}} =$ NL and $\mathrm{NL} = \mathrm{C_{=}L}$ in Chapter 2,
- $\star$ using the Isolating Lemma to show that $\mathrm{NL/poly} = (\mathrm{UL} \cap \text{co-UL})/\mathrm{poly}$ in Chapter 2,
- $\star$ by proving closure properties of $\sharp\mathrm{L}$ and $\mathrm{GapL}$, and using results from elementary number theory we prove many interesting closure properties of $\mathrm{Mod}_p\mathrm{L}$ and a characterization of $\mathrm{Mod}_k\mathrm{L}$ in Chapter 3, where $k, p \in \mathbb{N}$, $k \geq 2$ and $p$ is a prime,

- ⋆ the double inductive counting method used to prove a combinatorial closure property of $\sharp$L under the assumption that $\mathrm{NL} = \mathrm{UL}$ in Chapter 2 and its implications for ModL in Chapter 3, and
- ⋆ using polynomials to approximate the sign of a $\mathrm{GapL}$ function and its applications to show closure properties of PL in Chapter 4 such as the closure of PL under logspace Turing reductions.

In Chapter 5, we list a set of problems which are complete for logarithmic space bounded counting classes under logspace many-one reductions, all of which are based on the results that we have discussed in Chapters 2 to 4. In Chapter 5, we also define hierarchies of logarithmic space bounded counting complexity classes and complexity classes based on Turing reductions that involve Boolean circuits containing oracle gates for various logarithmic space bounded counting complexity classes. We show that these two notions coincide for all logarithmic space bounded counting complexity classes and as a consequence of the results shown in Chapters 2 to 4, some of the hierarchies also collapse to their logarithmic space bounded counting complexity class itself. In Chapter 6 of this monograph, we deal exclusively with one of the very important and useful theorems and its consequences on logarithmic space bounded counting classes that computing the determinant of integer matrices is logspace many-one complete for $\mathrm{GapL}$. We state and explain two very beautiful and deep theorems of M. Mahajan and V. Vinay on the determinant and also show many applications of the logspace many-one completeness of the determinant for $\mathrm{GapL}$ to classify the complexity of linear algebraic problems using $\mathrm{GapL}$ and other logarithmic space bounded counting complexity classes.

Since counting classes are defined based on non-deterministic Turing machines, invariably every theorem statement that intends to prove a property of a logarithmic space bounded counting class which we have covered in this monograph has at least one non-deterministic algorithm which has been either made explicit or explained in an easy to understand manner. We refrain from giving explicit algorithms for logspace many-one reductions since they are routine algorithms computed by deterministic Turing machines. We do not claim uniqueness over the order in which the chapters of this textbook is written since many theorems or statements proved in this monograph may have various proofs depending upon how we introduce this topic and order the results.

As a pre-requisite to understand this monograph, we assume that the student is familiar with computation using Turing machines and has undertaken a basic course on the Theory of Computation in which a proper introduction to complexity classes, reductions, notions of hardness and completeness, and oracles have been given.

I am extremely grateful to my doctoral advisor V. Arvind for the continuous encouragement and support he has given to me in pursuing research ever since I joined the Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai-600113 as a research scholar in Theoretical Computer Science. His invaluable guidance, ever encouraging words and timely help have rescued me from tough situations and helped me shape this research monograph. I am extremely grateful to

Jacobo Toran for a gentle proof reading of the results shown in this manuscript. His valuable comments and suggestions have significantly improved the presentation of the results shown in this monograph. I also thank Meena Mahajan for some useful discussions pertaining to the result that computing the determinant of integer matrices is logspace many-one complete for GapL.

I am extremely thankful to the higher officials of the Vels Institute of Science, Technology & Advanced Studies (VISTAS), Pallavaram, Chennai-600117 for their encouragement given to me in preparing this monograph. I also thank my colleagues in the Department of Computer Science and Information Technology, School of Computing Sciences, VISTAS and my friends for many enjoyable discussions while I was preparing this monograph.

I am extremely indebted to my mother for always being prompt and never late in cooking and providing me food everyday and in taking care and showing attention pertaining to any issue of mine if I lacked confidence and direction in tackling it. Any useful comments regarding this book may be sent by email to the email address `tcv.tclsbc@gmail.com`.

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