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Mathematics Conceptual Understanding and Pedagogical Frameworks : A Zimbabwean Perspective

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Published, marketed, and distributed by:

Deep Science Publishing
USA | UK | India | Turkey
Reg. No. MH-33-0523625
www.deepscienceresearch.com
editor@deepscienceresearch.com
WhatsApp: +91 7977171947

ISBN: 978-93-49910-49-2

E-ISBN: 978-93-49910-85-0

https://doi.org/10.70593/978-93-49910-85-0

Citation: Kazunga, C., Sunzuma G, Chiromo, L., & Zezekwa, N. (2025). *Mathematics Conceptual Understanding and Pedagogical Frameworks: A Zimbabwean Perspective*. Deep Science Publishing. https://doi.org/10.70593/978-93-49910-85-0

Preface

This book comprehending matrix operations is presented in Chapter 1. The chapter examines matrix addition, scalar multiplication, matrix multiplication, early genetic decomposition, and modified genetic decomposition. The initial genetic breakdown served as an analytical instrument. The study described in Chapter 2 was based on a paper that used Action Process Object Schema (APOS) theory to examine participants' beliefs of determinants and their attributes. Determinants of a matrix, determinants of transposition of a matrix, determinants of product of matrices, and determinants of inverse of matrix are shown in the early genetic decomposition. In order to determine the necessary determinants of matrices, certain questions call for the use of determinant properties. We examined the conceptual understanding of a solution of systems of equations served as the basis for the preliminary genetic decomposition presented in Chapter 3. The current study set out to ascertain the in-service math teachers' conceptual comprehension of the idea of solving systems of equations. The questions asked about the solution to equation systems and the geometric depiction of equation systems with consistent and inconsistent solutions using lines and planes.

The mental knowledge of students and recurring misconceptions regarding quadratic functions and equations are examined in Chapters 4 and 5. Chapter 4 t investigate how undergraduate students assimilate the idea of a quadratic function. With an emphasis on generalization, operational confusion, and interference, Chapter 5 examines typical mistakes and misunderstandings students make while attempting to solve quadratic equations. Both chapters draw attention to the cognitive difficulties that students encounter and stress the need of constructivist, student-centered methods in resolving these difficulties. When taken as a whole, these chapters provide important insights for enhancing secondary and tertiary mathematics education. Pedagogical models and framweworks were developed in this book.

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Deep Science Publishing
https://doi.org/10.70593/978-93-49910-85-0

Chapter 1: The conceptual understanding and instructional approaches to matrix operations among in-service teachers in Zimbabwe

1 Introduction

Matrix operations are critical in the study of linear algebra because it requires few prerequisites. Linear algebra is one of the pioneer mathematics courses that several university mathematics undergraduate students undertake, and students frequently find the course challenging. The shift from elementary to advanced mathematics itself, however, could be just as challenging as the subject matter. The definitions are critical mathematical processes of mental change; this change signifies a shift from describing to defining, from persuading to proving logically, and from the coherence of fundamental mathematics to the implications of advanced mathematics (Chaudhary, 2024 &Tall, 1991). "The teaching and learning of linear algebra at the university level is almost universally regarded as a frustrating experience," observe Sierpinska, Dreyfus, and Hillel (1999, p. 7). Studies in undergraduate linear algebra have consistently indicated that while students manage the procedural elements of the course, they struggle with grasping the conceptual foundations of linear algebra (Ozdag & Aygor, 2012; Plaxco & Wawro, 2015; Ndlovu & Brijlall, 2015). Although research highlighting these challenges has been conducted in various countries, there is a notable lack of studies specifically addressing students within the African context. This study examines various matrix operations, including “addition, scalar multiplication, linear combinations, matrix multiplication, and transposition” (Kazunga & Bansilal, 2017 p, 81). These operations are fundamental in linear algebra and have numerous applications in mathematics, science, and engineering. By understanding these operations, the study aims to contribute to a deeper understanding of matrices and their applications.

The study investigates the mental models of matrix operations held by 116 in-service mathematics teachers. The Action Process Object Schema (APOS) theoretical framework explains the development in comprehending mathematics concepts through the hierarchical growth of mental constructions called action, process, objects and schema.

1.1 Theoretical Framework

APOS theory was the main theory which underpinned this study. APOS theory is a constructivist theory focusing on the individual's mental constructions of mathematical knowledge.

APOS theory

APOS theory is a framework for learning complex mathematical concepts (Dubinsky 1996; Weyer, 2010). APOS builds on Piaget's "reflective abstraction and extends it to mathematics education" (Arnon,et al,2014 p.5). The "APOS theory involves concept formation from existing mental (or physical) actions" (Arnon, et al, 2014 p.17). It involves general descriptions of the mental structures and mental mechanisms (Arnon et al., 2014). Dubinsky (1991 p.19) pointed out five types of "mental mechanisms as interiorisation, coordination, reversal, encapsulation and generalisation" . These will lead to the construction of" hierarchal mental structures such as Actions, Processes, Objects and Schemas" (Ed et al., 2005 p.335).

Description of mental mechanisms

Arnon et al. (2014) described an action as an externally directed transformations of a previously conceived concept(s). The mental mechanisms as actions which are repeated and reflected on as the student moves from relying on external cues to having internal control over them (Arnon et al., 2014). The ability to mentally rehearse actions, scenarios, or processes. This involves creating an internal model of what would happen if certain actions were taken. "Interiorisation is the mechanism that makes this mental shift possible" (Engeness,2021 p.47). The ability to use symbols, language, and mental images to create internal processes (Wijayanti,et al., 2019 p.280) is indeed called interiorization. This process allows individuals to understand and make sense of perceived phenomena by constructing mental representations. For example, learning a mathematical concept involves initially acting on physical objects, then internalizing those actions into mental processes, and eventually encapsulating those processes into mental objects. The term encapsulation describes a "cognitive process where an individual perceives a dynamic process (like a system or software) as a static entity to which actions can be applied" (Arnon et al., 2014 p.21). This means the process is viewed as a container or object with predefined properties and behaviors, rather than a constantly evolving or interactive system.

Encapsulation occurs "when the individual applies an action to a process; that is, the individual student sees a process as a static structure to which actions can be applied. When a process has been encapsulated into a mental object, it can be de-encapsulated back to its underlying process when the need arises" (Chagwiza et al.,2020 p.3).This process is crucial for understanding and solving complex problems in mathematics education. The application of the mechanism of de-encapsulation, an individual can revert to the original process that gave rise to the mathematical entity. The mechanism of coordination is key in the development of some objects. Two objects can be de-encapsulated to form a new object.

1.2 Description of mental Structures

Action

An action is characterized as a physical or mental operation that transforms existing objects into new ones (Dubinsky, 1997; Weyer, 2010). Within the framework of Piaget's theory, as adapted by the APOS (Action–Process–Object–Schema) model, the initial understanding of a concept is formed through actions—that is, through externally directed operations applied to previously understood objects (Arnon et al., 2014). These actions are regarded as external because each step in the transformation must be carried out explicitly, often under guided instruction. The steps proceed in a linear, sequential manner, with each one prompting the next. At this stage of understanding, the process cannot yet be internalized or mentally simulated; the individual must perform each step in full to comprehend the concept (Arnon et al., 2014).

Process

A process is understood as a transformation applied to one or more objects, which the individual can control internally, without relying on external prompts. The individual is capable of mentally tracing or reflecting on the steps involved in the transformation without having to physically carry them out. Once constructed, a process can be manipulated in various ways—for instance, it can be reversed or coordinated with other processes (Dubinsky, 1991; Weyer, 2010).

Object

In many cases, a process functions under specific conditions. For example, when two objects are de-encapsulated, their corresponding processes can be coordinated and subsequently encapsulated to form a new object (Arnon et al., 2014). Consider the example of evaluating $(AB^T)^T$ to perform this operation correctly, the individual must de-encapsulate the matrices A and B^T to recall the process that defines them, then apply the necessary transformations, and finally encapsulate the result into a new cognitive object.

A condition refers to a particular value assigned to a property within a knowledge object. When this property takes on the appropriate value, the condition is satisfied, and the process can proceed. Conversely, if the property does not meet the condition, the process fails. For instance, the process is invalid if the individual mistakenly multiplies matrices A and B instead of the correct order or components (e.g. B×B).

Schema

A schema is a cognitive structure that encompasses the descriptions, organization, and exemplifications of mental constructs a student has developed regarding a mathematical concept (Arnon et al., 2014). For example, a schema related to systems of equations may include matrices as objects and integer operations as processes. This study will utilize these mental structures and the associated cognitive mechanisms to investigate students' understanding of matrix algebra concepts.

2 Literature review

The magnitude of students' challenges with the college linear algebra course has concerned numerous math education scholars worldwide (Dorier & Sierpinska, 2001). According to these authors, there are issues with the curriculum design of linear algebra, which makes the subject challenging both conceptually and cognitively. According to several experts, students' views of the challenges stem from the various ways in which they comprehend the topics. Similarly, Ndlovu and Brijlall (2015) observe that although students manage the course's procedural components, such working with matrices and solving linear equations, they have trouble grasping the fundamental conceptual concepts that underlie them. Since most mathematical problems need knowledge of specific techniques rather than a conceptual comprehension of the idea, many students occasionally perform fairly well on their final exams at the end of the linear algebra course (Siyepu, 2013). Nonetheless, Hiebert (2013) warns that it's critical to understand that the connections between procedural and conceptual knowledge evolve over time and are impacted by a variety of internal and external factors that affect the learner. According to Hiebert (2013), the connections between procedural and conceptual knowledge allow for the deconstruction of some mathematical paths that occasionally provide serious challenges, such as linear algebra. According to Star (2005), more research on deep procedural knowledge in mathematics education is necessary. He contends that the approaches used to evaluate students' procedural knowledge are insufficient because they just concentrate on their aptitude for solving mathematical problems. Since many concepts in linear algebra are given as procedures first, procedural knowledge should serve as the cornerstone of mathematical learning at all levels. Students are able to identify characteristics and connections between items that are included into the procedures as their comprehension of these processes grows. It is evident that more research is required to fully understand how students acquire their comprehension of fundamental ideas like matrix operations in linear algebra (Kazunga & Bansilal, 2016).

Maharaj (2015) used Action Process Object Schema theory in conjunction with instrumental and relational understanding to assess student teachers' comprehension of addition of matrices in linear algebra. He conducted interviews with two pupils to find out how they answered two tasks that included adding matrices made up of algebraic words. The findings showed that, although being able to subtract the matching elements, one of the students was unable to understand the equality of two matrices represented in the symbolic form. Additionally, the same student could not understand the property that multiplication of matrices is not commutative and was unable to expand matrices in algebraic form $(C + D)^2$. The student showed special difficulty communicating the equality link between two matrices using the symbolic language.Instead of utilizing the zero matrix, the student used the zero-number sign to represent the outcome of subtracting two equal matrices (Maharaj, 2015). The author emphasizes the symbolic notation's communicative role in mathematics and claims that students' comprehension of a concept is demonstrated by how well they use symbolic notation to convey pertinent mathematical ideas and relationships (Maharaj, 2015).

The results of a study conducted by Ndlovu and Brijlall (2015) with undergraduate students on mental constructions in matrix algebra support those of Siyepu (2013), who found that while most participants were comfortable using algorithms, they struggled to respond to questions that asked them to justify specific observations. According to other research, students complete procedures with ease; nevertheless, their low prior understanding of fundamental algebra hinders their ability to form the essential mental structures for matrix algebra (Ndlovu & Brijlall, 2015; Maharaj, 2015; Kazunga & Bansilal, 2015). By coining the term "met-befores" to refer to the prior experience, De

Lima and Tall (2008) emphasized the importance of understanding previously encountered concepts when learning new ones.They contend that when learning new concepts, prior experiences can lead to significant disagreements. According to Tall, De Lima, and Healy (2014), as helpful and challenging "met-befores" impact subsequent learning in increasingly complex mathematical situations, the gap between success and failure may widen. Additionally, Tall et al. (2014) recommend that current math teachers create a method that considers the ideas that every student has previously encountered.

3 Methods and materials

In this chapter we adopted an interpretive research paradigm, which recognizes that individuals' lived experiences and social contexts influence how they construct meaning (Guba & Lincoln, 2005; Henning, Van Rensburg, & Smit, 2005; Wahyuni, 2012). The interpretive approach prioritizes participants' subjective perspectives, making it particularly appropriate for exploring how in-service mathematics teachers understand matrix operations.

Aligned with interpretivist paradigm, the study utilized a qualitative case study design, which allows for an in-depth, contextually rich exploration of a bounded phenomenon (Gomm, Hammersley, & Foster, 2011; Nieuwenhuis, 2012). The "case" in this research was defined as a group of first-year in-service mathematics teachers at a Zimbabwean university. The bounded nature of the case allowed for a focused examination of participants' mental constructions related to matrix operations, including matrix addition, scalar multiplication, linear combinations, transposition, and multiplication. The primary aim was to capture and analyze the nuanced ways in which these teachers conceptualize fundamental matrix concepts insights that are crucial for improving instructional design in tertiary mathematics education.

Participants

The participants in this study were considered unqualified mathematics teachers, as their previous training no longer met the required standards. Despite being experienced educators, they were regarded as atypical students due to their lack of formal qualifications. To upgrade their credentials, they enrolled in in-service training programs offered by local institutions. These programs were part of a broader initiative, supported by international aid organizations in partnership with the Zimbabwean government, aimed at enhancing the qualifications of practicing teachers.

The structure of the program involved two intensive block teaching sessions per semester, allowing participants to complete the course over a three-year period. These sessions were scheduled during school and university holidays and ran daily from 8:00 AM to 6:00 PM. A total of 116 in-service teachers enrolled in a linear algebra course that covered matrix operations agreed to take part in this study.

Tasks

Five activities that focused on different facets of matrix operation made up the research tool. These challenges were designed utilizing the original genetic decomposition. The first question focused on linear combinations, which included scalar multiplication and matrix addition. Matrix multiplication was the basis for Questions 3, 4, and 5, whereas the transpose of a matrix was the topic of Question 2. During the two weeks of the December block session, the 116 participants learned matrix algebraic topics that involved matrix operations. During the second block session, which took place over the April holidays, the participants completed additional exercises and tests on matrix operations and other linear algebraic concepts in addition to reviewing the material that had been presented in December. During the April block session, the participants were given the five matrix algebra tasks.

Data collection

The 116 participants' written answers to the tasks and the 13 chosen participants' one-on-one interviews were used to create the data. To maintain anonymity, the participants were coded with tags such as "S1," "S2," and so on, with no bearing on the sequence. This allowed the data to be organized but prevented the responses from being associated with the original participants in publications.

Participants were chosen through the use of purposive sampling for semi-structured interviews that were both audio and video recorded. 15 people were first chosen to take part in the interviews; they included five individuals who scored highly, five who scored averagely, and five who scored below average on the written tasks. Two chose not to participate because it was voluntary, therefore 13 people (P1, P3, P18, P39, P43, P46, P60, P62, P86, P96, P97, P103, P111) answered the interview invitation. The interview questions were created as an extension of the written assignments. Additionally, participants were questioned further to learn more about some of the ideas that guided their written answers.

Data analysis

Coding as "C" for right answers, "0" for wrong responses, and "D" for no response was the initial step in the data processing process. After that, we conducted a thorough content analysis that allowed us to move beyond the binary coding (correct, incorrect). In order to respond to research question 1, participant errors were now discovered. This article uses pictures of six interviewees' written work to highlight some of the mistakes found and misunderstandings inferred. In order to supplement the findings, the interview transcripts were also examined. The preliminary genetic decomposition, which is included in the theoretical framework in the next section, served as the framework for the study Preliminary genetic decomposition for matrix operations

The formal definitions and constructions of scalar matrix multiplication, matrix addition , matrix transposition and matrix multiplications are provided in detail below, along with a discussion of their applications in various mathematical and real –world contexts. We drew upon the discussion by Arnon et al. (2014, p. 51) on examples of ‘what a genetic decomposition is not, to refine our genetic decomposition. This was done to avoid the common errors which can confound a sound description of a genetic decomposition with description of teaching sequence or mathematical description of a concept’. It should be noted that the genetic decomposition does not explicitly encompasses linear combinations of matrices, which constitute the primary focus of the first question in the research instrument. While a linear combination constitutes a conceptual schema involving the coordination of matrix addition and scalar multiplication, it is important to note that the genetic decomposition does not explicitly address linear combinations of matrices. Rather, the decomposition concentrates on the underlying operations –scalar multiplication and matrix addition- which form the basis for constructing the concept of a linear combination. Consequently, although these foundational operations are articulated within decomposition, the integrated concept of a linear combination is not directly represented, despite being the principal focus of the first item in the research instrument.

Matrices Addition

Action: At the action level, the learner performs isolated addition steps to compute individual entries of the resulting matrix –row by row or column by column without coordinating these steps into a coherent process or engaging with the underlying structure of matrix addition.

Process: The individual is able to anticipate the results of summing corresponding elements without relying on step-by-step computational procedures.Furhermore, at a more advanced conceptual level , the student can treat the addition of scalar multiples of matrices as a unified process foregoing the intermediate step of calculating the scalar multiple individually.

Object: The individual will understand and interpret the overall effect of matrix addition on any given $n\times m$ matrix. They will be able to explain when and why matrix addition is possible or not, based on the dimensions involved.

Additionally, the individual will be capable of applying further operations or transformations to the resulting matrix sums.

Scalar Matrix Multiplication

Action: The individual multiplies each element at a time by k, limited to an action conception. An individual cannot think beyond the single multiplication being carried out.

Process: An individual reflects on the rule and thinks about the effect of the scalar k on all the elements of the row or column or matrix A to form kA, by imagining that each element has been multiplied by the scalar k. The individual has interiorized the scalar multiplication and can carry out operations without doing step-by-step procedures. He or she is able to express the result of the scalar multiple symbolically using algebraic notation.

Object: The individual can see the effect of the scalar multiplication as a totality. The individual will be able to apply processes or further transformations on a scalar multiple of a matrix or scalar multiple of a row or column.

Understanding the Transpose of a Matrix

Action: The individual performs a single transformation of a row to a column, by systematically considering each row and transforming it into a column in a step-by-step manner without thinking beyond the rearrangement of each row.

Process: The individual is able to mentally visualize how converting each row of a matrix into a column changes its structure. They also understand that by reversing this process—turning columns back into rows—they can reconstruct the original matrix.

Object: The individual perceives the transpose operation as a complete transformation affecting the entire matrix. They can treat the transposed matrix (A^T) as a distinct mathematical object, apply additional operations such as matrix addition to it, and understand that performing the transpose operation twice returns the matrix to its original form.

Matrix Multiplication

While some students may have reached the object level of understanding, the analysis is limited to exploring only the action and process conceptions, which are outlined below.

Action: When computing the matrix product AB=C, the individual can perform the multiplication one step at a time by focusing on a single row from matrix A and a single column from matrix B. They multiply corresponding elements from the row and column, sum the results similar to computing a dot product—and place the result in the appropriate position. Specifically, they recognize that the i^{th} row of A and the j^{th} column of BBB together determine the c_{ij}entry in the resulting matrix C.

Process: The individual is able to imagine the effect of finding the dot product of the ith row of the first matrix with the ^{jth} column of the second matrix to generate a new specific elementc_{ij}. He or she does not necessarily have to go through the pair-wise multiplication of each element of the row with each element of the corresponding column but is able to recognize the corresponding elements of the rows and columns that are paired.

APOS insights emerging in this study

With respect to the concepts of scalar multiplication and addition of matrices, the content analysis and interview responses reveal that most participants had at least reached an action-level understanding. They could visually assess whether two matrices could be added and successfully carry out linear combinations when appropriate. Moreover, some interviewees were able to explain the operations of addition and scalar multiplication verbally without needing to perform each step explicitly, indicating a process-level comprehension. The preliminary genetic decomposition provided valuable guidance for the APOS analysis.

4 Results and discussions

The majority of participants demonstrated action-level engagement with transpose's operation, and interviews revealed that many had not progressed past this action conception. Because they had resorted to memorizing the rules by heart, data from interviews with some participants indicate that the majority of participants were still using action conceptions to reason.

We were able to predict the participants' levels of engagement with the matrix multiplication concept based on their answers to Questions 3, 4, and 5 as well as their interview responses. In Item 3.3, 90% of participants recognized that a (2 × 3) matrix could not be multiplied by a (2 × 2) matrix, suggesting that they might be acting at process level.However, only 78% of respondents were able to perform the matrix multiplication in Question 3, suggesting that some may have learned the rule by rote. Only 48% of participants were able to multiply a column matrix by a row matrix, according to the first section of question 3. This item, which involved multiplying a matrix with one column by one with one row, appears to have presented difficulties because the participants were not familiar with these matrix types. As a result, it's probable that some participants' process conceptions of matrix multiplication were incomplete.This is supported by their difficulties in Question 4, where they were asked to determine the order of AB and BA, if defined, after being given the order of matrices A and B using numbers (for instance, 2×3). As a process-level skill, this question asked participants to visualize performing the multiplication without explicitly going through each step. The fact that nearly half of the participants couldn't correctly answer all four questions suggests that they didn't have process conceptions of matrix multiplication.Their difficulties in answering Question 4, which asked for the order of AB and BA, if defined, after providing the order of matrices A and B using numbers (for instance, 2×3), support this. A process-level skill, this question asked participants to visualize performing the multiplication without explicitly going through each step. Nearly 50% of the participants failed to correctly answer all four questions, indicating that they lacked process conceptions of matrix multiplication.

The study also demonstrates that conclusions regarding APOS reasoning levels could not be drawn based only on whether or not participants provided accurate responses. Although one student's interpretation of the matrices with order 2×3 and 3×5 was incorrect, her answers to the order of AB in Question 4 first part and the order of BA in Question 4 third part were correct, suggesting that some members of the group may have given correct answers based on flawed reasoning.When given the order, S61 attempted to generate matrices, demonstrating that she needed to examine concrete matrices in order to determine whether the procedure could be completed. Similarly, it is not advisable to draw conclusions about participants' conceptions based solely on whether or not they answered an item correctly because it is possible that some participants may have had a process view but may have made a mistake.

Given that S96's work demonstrated that her arguments were backed up by the examples of matrices she generated, it appears that some participants required the convenience of a physical matrix in order to determine the order of the product matrix. To assist her in determining the matching row and column elements of the two matrices, S96 generated two general matrices of order 3×4 and 4×3. She may have been working on a process-level conception of matrix multiplication, but she still wanted to verify the alignment of the row and column elements, as evidenced by her use of general matrices, which allowed her to predict the order of the product matrix without having to perform every step of the multiplication.According to Tall (2004), mathematical thinking can progress from the embodied world through the world of symbols and finally to the formal world. The participants' varying success rates on the various questions imply that there is a difference in how the matrix is displayed. Regarding the answers to Questions 3, 4, and 5, the participants found that, when provided with concrete matrices (Q3), it was simple to determine the product's order. The success rate decreased when the order was provided in numerical form (Q4), and even fewer people succeeded when the order was provided in algebraic form (Q5). As a result, a large number of participants appeared to favor the embodied world (Tall, 2004).

According to a large number of participant responses, even at the action level, they were unable to interact with the matrix multiplication concept. This calls into question their methods in the classroom. Participants who have trouble understanding these ideas will not be able to identify the demands of questions or develop interventions that will meet the needs of their students. Because of their narrow perspectives, they are unable to identify their students' misconceptions, which may be misconceptions they themselves hold. Therefore, the pedagogic content strategies they will be able to use in the classroom are hindered by their limited understanding. Next, we offer revised genetic decomposition that takes into account some of the problems that this study brought to light.

We next present a revised genetic decomposition based on some of the issues that emerged in this study.

Revised genetic decomposition

Alright, here’s the deal: we went back to the drawing board and reworked the whole genetic decomposition thing using the latest analysis we did. This time, we tossed in a rundown of the must-know concepts for making sense of matrix operations—because, let’s be real, not everyone remembers that stuff. Also, we took the schema for linear combination and made the connection way clearer, tying it right to matrix addition and scalar multiplication. Oh, and we tweaked the way we talk about action, process, and object for each operation—so, if you check out Table 1, all those actions are actually spelled out clearly as in Table 1.

.Model of modified genetic decompositions

Matrix transformations

Prerequisite

Basically, the person looks at a matrix like it's a thing you can actually mess around with—add stuff, flip it, whatever. They can talk about what makes matrices tick, and they know how to actually do these matrix moves, not just in theory but, you know, really move things around.

Linear Combination of Matrices

Okay, here's the deal: when people talk about a "linear combination" of matrices, they're just mixing a bunch of matrices together using regular old addition and multiplying each one by whatever numbers they feel like (well, usually we call 'em scalars, because, you know, math people love fancy vocabulary). Nothing mystical. You take some matrices, slap a few numbers in front, add them up bam, that's your linear combo. Way less intimidating than it sounds, honestly.

Matrix addition

Action conception

You just eyeball the matrices and see if they're the same size—otherwise, forget adding them. If they match up, then hey, just add up each little spot with its partner. Easy. If not? No dice.

Process conception

So, imagine this—what starts off as just, you know, doing something, kinda turns into a full-blown process in your head. Like, instead of just slapping numbers together, you start seeing the whole "adding matrices" thing as its own step-by-step thing. At some point, you just know—almost instinctively—whether you can actually add those matrices or if the whole thing's gonna fall apart. It's like mental math morphs into a little routine you run without even thinking too hard about it.

Object conception

So, basically, the whole idea clicks into place when someone can look at any matrix, slap that transpose on it, and actually see the big picture—like, not just what happened, but why. If you can ramble off the quirks of a matrix transpose and mess around with it—flip it, tweak it, whatever—you're not just going through the motions. You actually get what's going on. That's when you actually own the concept, you know?

Matrix transpose

Action conception

The individual demonstrates the ability to transpose an $m \times n$ matrix, systematically interchanging its rows and columns to produce an $n \times m$ matrix. Furthermore, they can compare the transposed matrix to the original to determine whether the two are equivalent—recognizing, for example, that a matrix is equal to its transpose only if it is symmetric. This reflects an understanding of both the procedural and structural aspects of transposition.

Process conception

The action of transposing a matrix becomes interiorized into a process when the individual can mentally determine the result of a transpose operation without explicitly performing each step. At this stage, the individual recognizes the structural changes involved and understands that applying the transpose operation a second time reverses the effect, thereby returning the matrix to its original form. This indicates a conceptual grasp of transposition as a reversible process, rather than a sequence of isolated steps.

Object conception

The process of transposition becomes encapsulated into an object when the individual perceives the transpose not merely as a procedure, but as a complete and manipulable entity. At this stage, the individual can consider the transpose of a matrix as a whole, reflect on its structural properties, and articulate key characteristics—such as the fact that $(A^T)^T=A$,$(A+B)^T=A^T+B^T$and $(AB)^T=B^TA^T$. Additionally, they can apply further actions or transformations to the transpose itself, treating it as an object that can be analyzed, transformed, or embedded within more complex operations.

Multiplication of matrices by a scalar

Action conception

At the action conception of understanding matrix multiplication, the individual is able to compute each element of the product matrix by explicitly performing the necessary calculations. This involves identifying the corresponding row of the first matrix and the column of the second matrix, multiplying each pair of elements, and summing the results to obtain each individual entry in the product..

Process conception

The action of scalar multiplication becomes interiorized into a process when the individual can mentally anticipate the outcome without explicitly performing each individual step. At this stage, the learner can quickly determine the result of multiplying a matrix by a scalar by visualizing the effect—such as scaling each entry—without having to compute each multiplication separately. This demonstrates an internalized and conceptual understanding of scalar multiplication.

Object conception

The process of scalar multiplication is encapsulated into an object when the individual is able to represent and manipulate it symbolically using algebraic notation, such as αC, where αis a scalar and C is a matrix. At this stage, the learner can reflect on scalar multiplication as a mathematical entity and coordinate it with other processes—such as matrix addition or matrix multiplication—to explain and justify properties, including:

α(C+D)=αC+αDA+ (distributive property over matrix addition),

(α+β)C= αC+βC(distributive property over scalar addition),

α(β)C=(αβ)C (associativity of scalar multiplication).

This level of understanding allows the individual to reason flexibly about scalar multiplication and its interactions with other operations in matrix algebra.

Multiplication of matrices

Action conception

The individual demonstrates the ability to perform matrix multiplication involving at least two matrices. Specifically, they can identify the i^{th} row of matrix A and the j^{th} column of matrix BBB that must be multiplied to obtain the ij^{th}entry, $C_{ij}$of the resulting matrix C. Additionally, when presented with two matrices, the individual can determine whether multiplication is defined by checking the compatibility of their dimensions.

Object conception

At the process stage, the action becomes interiorized. The individual is able to mentally envision the outcome of the dot product between the ith row of the first matrix and the jth{column of the second matrix, resulting in the specific element Cij of the product matrix C without needing to explicitly perform each step of the multiplication. This indicates an internalized understanding of the operation. Furthermore, the individual can determine and articulate whether the multiplication of two given matrices is defined, without engaging in the actual computation. This shift reflects a more conceptual grasp of matrix multiplication and demonstrates the development of a coherent mental process.

Conclusions

This chapter is based on research where we examined the answers to given questions on matrix operations and interviews were done as follow up with 13 mathematics in-service instructors about the fundamentals of matrix operations. In this chapter discovered numerous common misconceptions, including the following: the use of equal sign as a do something; matrices and their transposes are taken as one single entity; the multiplication and subtraction are interchanged and it is impossible to discern between definition of a matrix and that for the determinant of a matrix. Given that student teachers are educators who instruct their students in these ideas, there is worry that the students would be exposed to these misunderstandings while learning the material.The study also investigated the teachers' conceptual frameworks for matrix multiplication utilizing APOS theory, transposing a matrix, and linear combinations of matrices. According to the results, many participants found it difficult to answer interview questions and things that called for higher degrees of engagement with the matrix processes, but they were able to handle those that required action-level participation. It is clear that the participants required additional chances to interact with the ideas, which might have been restricted by the program's delivery schedule. Since they teach these subjects to their schoolchildren, it is anticipated thatin future, programs will attempt to include additional possibilities for participants to acquire the necessary information.

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Deep Science Publishing
https://doi.org/10.70593/978-93-49910-85-0

Chapter 2: Unpacking In-services Mathematics Teachers' Understanding of Determinant of Matrices

1 Introduction

The determinant of matrices is a scalar quantity which is used in many topics in Zimbabwean mathematics high school curricula. Usually taught in first-year university linear algebra courses for mathematics undergraduates. While determinants are commonly interpreted as scalar values for square matrices, they serve multiple important functions in linear algebra.. This may account for the inclusion of determinants in high school curricula in many countries, where students are introduced to foundational linear algebra concepts early on.. In the Zimbabwean ordinary level curriculum, for example, 'students learn determinants of 2 by 2 matrices, using inverse methods to solve simultaneous equations (systems of equations with two unknowns) and solving problems involving unknown for a singular matrix' (Zimbabwe General Certificate of Education Mathematics, 2012-2017, p.12). The Zimbabwean advanced level curriculum involves 'finding the determinant of n by n matrices, using inverse methods to solve simultaneous equations (systems of equations with n unknowns) and solving for the unknown for a singular matrix' (Zimbabwe Advanced Certificate of Education Mathematics, 2013-2017, p.20).

As educators with Mathematics teaching experience in Zimbabwe and South Africa, we observed that most high school students performed poorly on questions involving the determinant of matrices. Similarly, while teaching undergraduate mathematics at state universities, we have noticed comparable difficulties among students with this topic determinant of matrices.. The performance was poor with pass rate we have noticed that students' performance on this topic is usually far below the university's minimum expected pass rate of 75% each semester.On top of that, university lecturers agreed that students generally perform poorly on determinants and their properties.. Moreover, mathematics education researchers around the world have expressed concern about the challenges students face in undergraduate linear algebra courses.(Kaznga &Bansilal. 2017, Dorier & Sierpinska, 2001).

The agreement is that teaching matrix algebra and even linear algebra is a frustrating experience for both instructors and students. Despite ongoing efforts to improve the curriculum, learning linear algebra continues to be challenging for most students (Dorier, 2000; Sierpinska, 2000; Dorier & Sierpinska, 2001). These concerns led us to investigate the possible causes and the extent of students' struggles with problems involving the determinant of matrices. For teachers, developing a solid conceptual grasp of the determinant of a square matrix and its associated properties is crucial, as this knowledge underpins their ability to teach the topic effectively and respond to student misconceptions.

This study investigated the conceptual understanding of determinants among 116 student teachers enrolled in an undergraduate degree program in mathematics education. These teachers, who currently instruct at the ordinary level equivalent to grade 11, are undergoing professional upgrading to qualify for teaching at the advanced level – slightly above grade 12 of high school mathematics. The focus of the study is on their *mental conceptions* of determinant of a matrix, as interpreted through the lens of APOS theory—*Action, Process, Object,* and *Schema*. According to this framework, concept development is hierarchical: learners first form an action-level understanding, which may evolve into a process conception, then into an object conception, and finally into a coherent schema. However, this progression is not always linear, as individuals may fluctuate between levels or skip stages entirely.

This research addresses the question: *How does the APOS framework illuminate in-service teachers' understanding and conceptualisation of the determinant of a matrix?*

2 Literature review

The concept of the determinant is foundational in linear algebra and is approached from multiple perspectives within mathematics education. Traditionally, it is introduced to students as a numerical value computed from an n by n matrix. Beyond this, the determinant is also understood in geometric terms—representing the area of a parallelogram in two dimensions or the volume of a parallelepiped in three dimensions (Todorova, 2012; Donevska-Todorova, 2014). While these interpretations are valuable, they often overshadow a more abstract and formal view of the determinant of a matrix: as a function that maps an n by n matrix to a scalar while satisfying specific algebraic properties.

The determinant of a matrix should be introduced and understood as a function possessing key properties such as multilinearity, alternation, and normalization(Donevska-Todorova, 2016). These functional properties provide a deeper conceptual grounding and support a structural understanding of linear transformations. However, this functional perspective is rarely emphasized in school or even undergraduate curricula. Interpreting the determinant solely as a computational tool or geometric measure may result in a fragmented or superficial understanding of the concept (Todorova, 2014).

The concept of the determinant plays a fundamental role in linear algebra and has a wide range of applications. It is commonly used to solve systems of nnn linear equations with nnn unknowns and is central to the Invertible Matrix Theorem, which characterizes conditions under which a matrix is invertible. Additionally, determinants are useful in vector spaces, where they help determine whether a set of vectors is linearly independent. Beyond these foundational uses, the determinant is also instrumental in the computation of eigenvalues and eigenvectors—concepts that are essential in multivariate statistics, quantum mechanics, and the analysis of non-linear differential equations. Larson, Zandieh, and Rasmussen (2008), as well as Rasmussen and Blumenfeld (2007), refer to this method of introducing ‘eigenvalues and eigenvectors’ through the determinant as the *eigenvector-first approach*, which emphasizes the conceptual structure of eigentheory early in instruction.

The concept of the determinant extends beyond pure mathematics and plays a significant role in various scientific disciplines, including physics and computer science (Todorova, 2012). Determinants form a foundational component of scientific curricula at both secondary and tertiary levels and are often considered central to the study of quantitative sciences. Historically, the determinant emerged in the context of solving systems of linear equations. In 1750, the Swiss mathematician Gabriel Cramer developed a general method for solving systems of nnn linear equations with nnn unknowns. His approach involved the combinatorial manipulation of coefficients, which he notated using superscripts—though the variables themselves remained unspecified. This method, now known as **Cramer's Rule**, provided conditions under which a system of equations has a unique solution or none at all (Andrews-Larson, 2015).

Cramer’s notational system and structural approach to organizing coefficients significantly influenced the formal specification of the determinant. His work laid the groundwork for later developments in linear algebra. In the mid-19th century, English mathematician James Joseph Sylvester expanded on these ideas. In 1850, Sylvester introduced the term *matrix* and further advanced the use of determinants within matrix theory. As such, determinants not only serve as computational tools but also function as diagnostic instruments for analyzing the solvability and consistency of linear systems.

Matrix algebra, encompassing the determinants of square matrices, constitutes a fundamental topic within linear algebra that requires thorough comprehension (Bolgomony, 2007).A comprehensive understanding of matrix algebra involves not only the execution of calculations but also a deep grasp of the underlying theoretical concepts.It involves understanding how procedures work, anticipating results intuitively, adapting algorithms, and recognizing connections within experiences (Bolgomony, 2007).

Globally, undergraduate students face significant challenges in mastering linear algebra, particularly in developing a robust conceptual understanding. Studies by Stewart and Thomas (2009), Possani, et.al (2010), Wawro (2011), and Ozdag and Aygor (2012) have documented these difficulties, attributing them primarily to the abstract nature of the subject. Students' previous mathematical knowledge, especially in structures and set theory, often falls short in providing a strong foundation for constructing new knowledge in linear algebra (Dogan, 2011). Similarly, students struggle not only to understand but also to explain and interrelate the theoretical concepts they learn(Carrizales 2011). This body of research underscores the idea that many linear algebra concepts are inherently complex and intangible, posing significant obstacles to student comprehension.

Ndlovu and Brijlall (2016) argued that mathematics educators should shift their focus from procedural fluency to fostering students' conceptual understanding of the relationships between mathematical ideas. They highlight that students tend to assimilate procedures—such as those involved in calculating determinants—as a series of disconnected actions, which limits deeper comprehension. In light of this, the present study explores how Zimbabwean student teachers understand the concept of determinants and their associated properties. Ndlovu and Brijlall (2016) further stressed that insights into students' mental constructions and their interconnections are vital for developing instructional strategies that support meaningful learning. Todorova (2016) reinforced this viewpoint by demonstrating that many students struggle with the concept of multilinearity, frequently failing to apply its formal definitions correctly when working with determinants of square matrices.

3 Methods and materials

This chapter draws from a broader study examining how student teachers understand linear algebra concepts (Kazunga & Bansilal, 2017a; Kazunga & Bansilal, 2017b). The study followed an interpretative approach, recognizing that individuals construct knowledge based on their diverse experiences (Guba & Lincoln, 2005; Wahyuni, 2012). It involved qualitative analysis of written answers from 116 Zimbabwean teachers responding to six questions about determinants of matrices and their properties. These teachers were enrolled in an introductory linear algebra module within a three-year in-service programme, covering matrix algebra, determinants, and solving linear systems. The determinant was approached as a numerical attribute of square matrices. The programme was structured so that the content of a typical three-year undergraduate degree was delivered over three years through intensive block-release sessions held twice each semester during holidays, with classes running daily from 8 a.m. to 6 p.m.

Data were gathered from individual activity sheets, which were marked by the first author. Five questions assessed the teachers' competence in calculating determinants and understanding their properties. After analyzing the responses, 15 teachers were chosen for individual interviews to better understand their thought processes. The group included five high performers, five average performers, and five below-average performers. Two participants did not attend, so 13 semi-structured interviews were conducted to explore their experiences and perspectives (Guba & Lincoln, 2005). A flexible interview guide focused on determinants was used, and interviews were audio- and video-recorded. Ethical considerations were upheld, ensuring anonymity through pseudonyms. Preliminary genetic decomposition was employed as an analytical framework to interpret data from both written work and interviews.

Theoretical Framework

In research on students' conceptual understanding of mathematics, APOS theory—standing for action, process, object, and schema—has proven to be a valuable framework. This constructivist theory describes the cognitive pathways individuals follow as they develop mathematical concepts, emphasizing the mental constructions involved in learning (Dubinsky, 1997; Weyer, 2010; Arnon et al., 2014).

According to APOS theory, the mental constructions of action, process, object, and schema are hierarchical. First, a learner develops an action conception, which then can grow into a process conception, and eventually into an object conception. An action involves any physical or mental transformation of objects to obtain new objects (Dubinsky, 1997; Weyer, 2010). A process is an internal mental operation on objects that the learner controls independently. This process can be manipulated—reversed or combined with others. When a learner recognizes this process as a whole and can explicitly work with transformations on it, the process becomes an object in their cognition. Many studies (e.g., Tall, 1991; Parker, 2010; Dubinsky, 1997; Dorier, 1990; Carlson, 1993; Cowen, 1997; Sabella & Reddish, 1995; Cooley et al., 2007; Possani et al., 2010; Carrizales, 2011; Matthews, 2012) have used APOS theory to explore how students understand linear algebra. Dubinsky (1997) called for more research to identify the mental constructions students form when learning linear algebra concepts and to develop teaching methods that support these constructions. Matthews (2012) pointed out that solving one linear equation is easier than solving multiple equations, which is important for designing effective teaching strategies.

This framework is particularly relevant for studying the learning of determinants, a concept that often challenges students due to its abstractness and the procedural emphasis found in many instructional approaches (Todorova, 2012; Donevska-Todorova, 2014). By using APOS theory, this chapter investigated how student teachers mentally construct the concept of the determinant, including the extent to which their understanding aligns with different APOS levels.

In doing so, the chapter responds to calls by Dubinsky (1997) for research into the specific cognitive structures students form when learning linear algebra, and how pedagogical strategies can support the development of appropriate mental constructions. APOS theory thus serves as both an analytical tool and a pedagogical guide for exploring conceptual understanding in linear algebra among participants. The following is a a model for the preliminary genetic decomposition of determinant of a matrix:

Prerequisite

A foundational expectation is that the learner can identify the size and order of a matrix, and has developed a conceptual (object-conception) understanding of key operations such as transposition and inversion of matrices.

Action conception

At this stage of understanding, the individual can identify whether the determinant of a given matrix is computable and apply a specific rule to calculate it, performing each step sequentially.

Process conception

At the process conception, the student is able to internalize the steps involved in evaluating the determinant, carrying out the procedure mentally without the need to explicitly articulate each action.

At this stage, the individual demonstrates the ability to:

- determine the computability of a matrix's determinant based on its dimensions;
- utilize determinant properties to evaluate matrices efficiently; and
- handle matrices containing algebraic expressions in the computation of determinants.

Object conception

At this level of understanding—aligned with the **object conception** in APOS theory—the individual is able to treat the determinant as a mathematical object that can be manipulated, compared, and related to other determinants through various operations. Specifically, the individual:

- can distinguish between determinants of related matrices and articulate the relationships between them;
- can apply operations to |A||A||A| in order to determine values such as |AT||A^T||AT| and |A−1||A^{-1}||A−1|, and explain the relationships among these determinants;
- is capable of applying actions informed by the 'multilinearity property' of determinants to solve problems.

  For example, given if $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix}$, then find $|A|$,

  given that $|B| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = n$, ,

  the individual can deduce that |A|=3|B|=3n, using the scalar multiplication property of determinants;
- understands and can explain the equivalence of computing the determinant via cofactor expansion along different rows or columns.

4 Results and discussions

Evidence of action conception

The APOS analysis in this study was guided by a preliminary genetic decomposition of the determinant concept, developed to explore the in-service teachers' mental constructions. In relation to the research question—*What insight can an APOS analysis reveal about in-service teachers' conceptualisation of the determinant concept?*—the findings offer important implications. Specifically, for square matrices of order 2, 3, and 4, the results from the written responses and interviews suggest that a majority of participants had developed at least an action conception of the determinant. This is evidenced by their ability to identify, often by inspection, whether a matrix was suitable for determinant computation, and by their step-by-step application of known algorithms for matrices of lower order.

However, some participants exhibited confusion when attempting to extend the 2 by 2 determinant rule to 3 by 3 or 4 by 4 matrices. These incorrect generalisations suggest that such participants had not yet fully constructed an action conception for higher-order matrices and were instead operating at a pre-action level, relying on memorised procedures without conceptual understanding.

The evidence further shows that over 50% of the participants were comfortable performing determinant calculations up to order 3, which reflects the internalisation of actions into routine procedures—hallmarks of the action level in the APOS framework. These findings resonate with those of Kazunga and Bansilal (2015), whose study similarly reported that full-time pre-service teachers were generally able to carry out determinant computations at the action level. However, the limited progression beyond this level, especially in applying determinant properties or reasoning structurally about higher-order matrices, suggests a need for pedagogical interventions that promote advancement toward process and object conceptions.

Thus, this analysis not only reveals the participants' predominant positioning at the action level of understanding but also points to the cognitive gaps that hinder their progression within the APOS framework. Addressing these gaps is essential for helping in-service teachers develop the deeper conceptual understanding necessary for teaching advanced school mathematics.

Evidence of process level reasoning

An analysis of the responses to Questions 2, 3, and 4 used for this chapter revealed that approximately 25% of the student teachers consistently provided correct answers, suggesting that these participants had likely interiorized actions into processes—as described in the APOS framework. These individuals demonstrated the ability to evaluate the determinants of square matrices with algebraic entries, and were able to predict whether the determinant of a matrix could be computed, indicating a more flexible and generalized understanding of the concept. Furthermore, some participants were able to describe the procedure for finding the determinant of a 3 by 3 matrix without explicitly performing the computations, further supporting the claim that they had reached the process level of understanding.

However, the responses to one of the questions exposed conceptual weaknesses in a significant portion of the cohort. Several participants were unable to determine the determinant of the inverse of a matrix, the transpose, or the product of two matrices when provided with only the determinants of the individual matrices (without numerical entries). These cases highlight a failure to internalize the properties of determinants, particularly the multiplicative and inverse properties. A number of participants made reasoning errors, such as incorrectly applying rules from logarithmic operations (e.g., treating $|A^2|$ as $2|A|$, analogous to $\log(m^n)=n\log(m)$. Such errors indicate that these participants had not yet developed the appropriate mental constructions needed to understand the formal properties of determinants.

In APOS terms, these participants appear to be functioning at the action level or even at a pre-process level with respect to determinant properties. They could apply memorised procedures in straightforward numeric cases but struggled to transfer these actions to abstract or symbolic contexts, suggesting a lack of schema integration. This finding highlights the importance of pedagogical strategies that support the transition from action to process, and ultimately to object-level understanding, where learners can view determinant-related operations as coherent, manipulable entities.

These results address the research question by illustrating the extent to which APOS theory can uncover variability in conceptualisation among in-service teachers, and by identifying specific cognitive hurdles that hinder the development of a deeper understanding of determinant properties.

Evidence of object–level reasoning

With regard to the application of determinant properties, most participants did not demonstrate evidence of object-level engagement as defined by the APOS framework. Both the written responses and interview data suggest that the majority of student teachers remained at the action level, where they were primarily able to recall and state determinant properties without deeper conceptual insight. Only a small number of participants were able to articulate the relationships between $|A|$, $|A^T|$, and $|A^{-1}|$ given the value of $|A|$, or to apply the multilinearity property to solve abstract problems, such as those presented in Question 3. Similarly, few participants were able to explain the equivalence of different determinant evaluation methods, such as expansion across various rows or columns.

Those participants who successfully solved items in both Questions 3 and 4 demonstrated greater flexibility in their reasoning. These individuals appeared to be approaching an object conception of determinants, as they could operate on the process of computing determinants and apply properties like multilinearity in a meaningful way. However, their responses to Question 5, which required the construction of a counterexample to disprove a general statement, revealed limitations in their reasoning. Many of these participants struggled to formulate original examples without relying on classroom-taught procedures, indicating that full encapsulation of the process into an object had not yet occurred.

Notably, only eight participants were able to construct valid counterexamples and justify why the given statement did not hold in all cases. These individuals demonstrated object-level understanding, as they could manipulate determinant concepts as mental objects and reason about their properties abstractly. This conclusion was corroborated by interview data, particularly in the case of one participant (pseudonym "John"), who provided clear evidence of object-level reasoning during follow-up discussions.

In this study, Question 5 served as a key diagnostic item that distinguished between participants who had reached the process level and those who had progressed to the object level. Approximately 25% of the participants exhibited process-level reasoning, yet lacked the structural understanding necessary for object-level conceptualisation. This finding suggests a need for targeted pedagogical interventions that can support the transition from process to object. For example, incorporating tasks that require learners to construct logical arguments or refute generalisations—such as those in Question 5—may provide the cognitive conflict needed to prompt re-evaluation of existing process conceptions. In doing so, participants may begin to encapsulate these processes into coherent mental objects, thereby deepening their conceptual understanding.

Conclusions

From this chapter, we conclude that most student teachers enrolled in the programme, who are currently teaching Ordinary Level mathematics in schools, demonstrate predominantly action-level reasoning regarding the concept of determinants, with some even operating at a pre-action level. This finding highlights a critical gap in conceptual understanding. It is imperative that mathematics educators seek innovative and effective pedagogical strategies that promote deep conceptual understanding rather than merely procedural fluency in determinants and their properties.

Future professional development courses for student teachers should provide ample opportunities for meaningful engagement with the concept of determinants to facilitate cognitive growth beyond the action stage. The application of APOS theory in this study proved valuable in identifying the specific cognitive levels at which the participants are functioning. The evidence suggests that while the majority operate at the action level, a minority are at the pre-action level, and only a very few have attained the object conception.

This situation is concerning, given that these teachers are responsible for teaching these mathematical concepts to learners. To be effective educators, teachers must possess a robust and well-developed schema of the mathematical content they deliver. Strengthening the conceptual understanding of in-service teachers is therefore essential for improving the quality of mathematics education at the secondary school level.

This chapter again revealed that most student teachers did not develop a deep conceptual understanding of determinants when the concept was presented solely as a numerical value. Therefore, mathematics lecturers and educators are encouraged to incorporate the historical development of determinants into their teaching, as this contextualization can enhance students' conceptual grasp (Larson-Andrews, 2015). Additionally, some scholars advocate for the use of dynamic visualization tools, such as the dimensional GeoGebra system, to aid students in exploring determinant properties. Visualizing how determinants change as the coordinates are scaled or transformed may facilitate discovery and internalization of key properties.

However, visualization and historical context alone may be insufficient for developing a robust understanding of determinants. Dorier (2000) argues that a 'practical' approach, which emphasizes real-world applications, is more suitable than a purely theoretical perspective, particularly since many university students tend to be practically oriented. Furthermore, a 'structural' approach that focuses on the underlying relationships and frameworks within linear algebra is especially advisable. By engaging with practical problems and applications, students are more likely to develop an object-level conception of determinants.

Mathematics lecturers could therefore enrich their instruction by integrating the history of determinants to spark interest and motivation, while also employing visualization technologies and emphasizing practical applications. Such a multifaceted approach may better support the cognitive transition from procedural fluency to deeper conceptual understanding, aligning with the goals of fostering object-level mental constructions of determinants.

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Deep Science Publishing
https://doi.org/10.70593/978-93-49910-85-0

Chapter 3: Conceptual Understanding of Application of determinants in finding solutions to systems of linear equations

1 Introduction

The application of the concept of determinants is apivotal subject in the study of linear algebra, other branches of mathematics and other science subjectsIt forms the basis for many critical concepts in linear algebra. The chapter concentrated on participants who teaching the topic concepts to their students which they were studying at a particular university. The chapter's purpose was to investigate the conceptual understanding of application of determinant of a square matrix infinding solutions to systems of linear equationsfor 116 student teachers.

Determinant though taught as a numeric value in most universities in developing countries play a vital part in solving system of equations. Todorova (2012) looked at conceptual difficulties that high school students have during the time when student learnt and are taught linear algebra and geometry. The author identified students' difficulties in understanding the concept definitions and various concept images of a determinant. Todorova (2012) asserts that through teaching experience the students sometimes do not build effective concept images in Linear Algebra and Analytical Geometry. The author recommends using GeoGebra to help students better understand what a determinant means. With this tool, students can actually see how the determinant is linked to the area of shapes on a plane. The author also explains that a 3×3 matrix's determinant isn't just about area—it can also be used to find the volume of 3D objects.

Students are generally able to understand the concept of a determinant as the value associated with a two- or three-dimensional matrix, or as representing the area of geometric figures such as squares, rectangles, and parallelograms, as well as the volume of three-dimensional shapes like cubes, boxes, and parallelepipeds (Todorova, 2012). However, their difficulties often arise when they need to apply the definition of a determinant to solve problems involving non-standard or abstract geometric figures. Todorova also notes that most textbooks introduce the concept clearly, and students typically have little or no trouble grasping that a determinant is a numerical value and learning the procedure for calculating it.

Students are capable of solving simple problems, even following the introduction of the formal definition.The students start to face difficulties when they are asked to solve problems that involve non-regular polygons such trapezoids, triangles, pentagon or a convex (or even concave). The study will explore the conceptual understanding of application of determinant in solving systems of equations. The research questions guiding this study are: 1) How do the student teachers perform on selected assessment items involving application of determinants in solving systems of equations? 2) What do written and interview responses of the student teachers reveal about the engagement conceptual understanding levels (using APOS theory) of student teachers with the concepts? It is anticipated that recognizing these trends will help guide the design and implementation of courses in similar contexts.

2 Literature review

The 'theory of determinant in linear algebra' emerged as result of finding solution to a systems of linear equations (Andrews-Larson, 2015, p10). 'Efforts to comprehensively characterise linear systems of equations and their solutions grow into the theory of determinants' (Andrews-Larson, 2015, p10). Andrews-Larson (2015) examined use of history to enlightenteachingsystems of linear equations. The author examined how instruction and instructional design in linear algebra can be enhanced by considering the subject's historical origins.

The theory of determinants emerged independently in both Japan and Europe between the 1600s and 1700s, following earlier methods developed in ancient China for solving systems of equations (Andrews-Larson, 2015). In 1693, Japanese mathematician Seki Kowa developed a version of the determinant as part of a method for solving nonlinear systems of equations (Kazunga & Bansilal, 2015, 2018).

In 1750, Swiss mathematician Gabriel Cramer independently developed a method for expressing the solution to a system of linear equations as a closed-form system. He generalized a technique for computing these solutions by utilizing combinatorial principles and cleverly arranged, though unspecified, superscripted coefficients. Cramer did not explain how his method was derived; instead, he presented a general rule for solving systems, framed in terms of the combinatorics of superscripted coefficients. To solve an n×n system, his method involves forming n fractions, each containing n! terms in both the numerator and the denominator.Cramer’s comments identify what the value of his denominator (i.e. the determinant) reveals about uniqueness of the solution set as to a square system of equations. The term "matrix" was coined in 1850 by the English mathematician James Joseph Sylvester, who worked extensively with determinants. Determinants, in this context, serve as a tool to determine whether a system of equations has a unique solution.

Determinants are numerical values derived from square matrices and are fundamentally linked to the calculation of area in two-dimensional spaces and volume in three-dimensional contexts (Todorova, 2012; Donevska-Todorova, 2014). They are essential in linear algebra, particularly for solving systems of equations, assessing matrix invertibility, and determining the linear independence of vectors.

Dorier and Sierpinska (2001) wrote on the obstacles of formalism in linear algebra. Students experience difficulties in linear algebra which they term formalism obstacles. Commonplace difficulties are due to formal manipulations, also insufficient background in logic and elementary set theory can lead to errors in the understanding and application of linear algebra concepts. Students condemn linear algebra at tertiary level because of many new terms, notions and absent of connections with what they know in linear mathematics. Many students feel like they have landed on unfamiliar environment and fail to get their way in this new sphere.Students’ difficulties in linear algebra represent a persistent and widespread challenge that has endured across successive generations and various teaching methods. They call it formal obstacle. Commonplace difficulties are due to formal manipulations absent prior knowledge on concepts of set theory contribute to the production of errors in linear algebra. All mathematics is characterised by a certain degree of formalism. The teacher’s task is to facilitate the students’ passage from one level of conceptualisation to the following one, as set out in official syllabus. When solving system of equations many calculation errors were often associated with methodological errors.

Many researchers have investigated the difficulties undergraduate students face when learning linear algebra. These challenges often stem from factors such as the abstract nature of the subject and the structure of the curriculum (Dorier, 2000; Sierpinska, 2000).The students’ challenges are called ‘formalism obstacle’ (Dorier & Sierpinska, 2001). The formalism of language, new notation, and elementary logic students experience as they learn determinant of a matrix concept are the obstacles they should conquer. The students’ deficiency in practice and competence in linear algebra concepts are the other cause of obstacles experience (Dorier *et al.*, 2000).

Mathematics educators and teachers should prioritize helping students understand the interrelationships between concepts, rather than focusing solely on procedural tasks (Ndlovu & Brijlall, 2016). In many schools, students tend to construct mathematical knowledge as isolated facts, often internalizing rules as a series of disconnected actions. The study aims at mental construction that in-service teachers are able to make when learning determinants.Determinants of matrices are an essential component of matrix algebra and play a key role in solving systems of equations. They are also included in the mathematics curriculum at high schools in developing countries. Ndlovu and Brijlall (2016) emphasize the importance of understanding the nature of students' mental constructions and how these contribute to the development of conceptual understanding in mathematics. This understanding is crucial for designing and implementing effective instructional strategies. They argue that pedagogy should focus on helping students build meaningful and relevant mental structures.The finding of the study will highlight challenges faced and also potentialapproaches that can be used to upgradedeterminants understanding. This may lead to more current teaching of determinants as mathematics educators know the mental conception students have of the topic and therefore, construct the teaching in a way that will help assist students develop the necessary skills and knowledge.

As one solves system of equations involving the methods which use determinant of a matrix, notation plays a vital role. MacGregor and Stacey (1997) observed that students are often unaware of the inherent consistency of mathematical notation and the power it offers in understanding mathematical concepts. Such misinterpretations can hinder students' ability to make sense of algebra and may persist for years if not identified and addressed.

Ndlovu and Brijlall (2016) analysed written work of 31 teachers on training and then selected 5 of them for interviews. They observed that few teachers on training were operating on process conception. These teacherson training understand the procedure of evaluating determinants to a point that they could explain the connection made between general statement of evaluating determinant and its applicability to other contexts. While Ndlovu and Brijlall (2016) focused on the use of Cramer's Rule for solving systems of equations among full-time pre-service teachers in South Africa, the present study explores both Cramer's Rule and the inverse matrix method as applied by part-time student teachers in Zimbabwe.

As noted by Dikovic (2007), the use of technology in teaching systems of equations helps foster deeper student understanding. It empowers students to tailor their learning journey, offering flexibility in choosing problems, generating examples, and engaging with topics at their own pace and interest level.

3 Methods and materials

This research forms a subset of a broader study that examined student teachers' understanding of linear algebra concepts (Kazunga & Bansilal, 2017a; Kazunga & Bansilal, 2017b).To carry out the study, a range of materials and tools were employed, which are described below.

Research Design

This study was guided by the interpretive research paradigm, which recognizes that individuals bring unique backgrounds and experiences that continuously shape the reality within their contexts (Guba & Lincoln, 2005; Henning, Van Rensburg & Smit, 2005; Wahyuni, 2012). The paradigm emphasizes the importance of understanding phenomena through the participants' own perspectives and definitions, particularly regarding matrix operations concepts (Henning et al., 2005). Our research focused on gaining a comprehensive understanding of how student teachers apply determinants in solving systems of equations. We adopted a case study approach—a systematic inquiry aimed at detailed description and analysis of a bounded system—to explain the student teachers' conceptual understanding of matrix operations (Gomm, Hammersley, & Foster, 2011; Nieuwenhuis, 2012). Identifying the specific unit of study and clearly delineating its boundaries is fundamental in case study research; here, the unit consisted of first-year student teachers at a particular university.

Participants

The study involved non-traditional participants—mature teachers categorized as unqualified mathematics teachers, given that their initial training was no longer considered sufficient. These teachers participated in a large-scale Teacher Capacity Program funded by international aid organizations in collaboration with the Zimbabwean government. The program aimed to upgrade their qualifications through in-service courses offered at local universities. It was designed to be completed over three years, with lectures delivered in two intensive block sessions each semester. These sessions occurred during school and university holidays and consisted of full-day classes from 08:00 to 18:00. A total of 116 teachers, enrolled in a linear algebra course focusing on matrix operations concepts, consented to participate in the research.

Data collection

The study generated data through the written responses of 116 student teachers and individual interviews with 13 selected student teachers. To maintain anonymity, participants were assigned codes (e.g., 'S1', 'S2') that were randomly ordered and held no significance. This method ensured that responses could not be linked to specific individuals in any published work, while aiding in the systematic arrangement of data.

Using purposive sampling, 15 participants were selected for semi-structured interviews, which were recorded both audio-visually. These participants were chosen to represent a range of performance: five scoring high, five average, and five below average on the written tasks. Since participation was voluntary, two opted out, resulting in 13 interviewees. The interview questions were crafted as follow-ups to the written activities, with additional prompts used to gain insight into the reasoning behind their written responses.

Data Analysis

Initially, data were coded using 'C' for correct responses, '0' for incorrect, and 'B' for blank answers. An in-depth content analysis then extended the coding beyond this simple classification. This deeper analysis allowed for the identification of errors made by participants, aimed at answering research question 1. The article includes images of written work from six interview participants to highlight some of the observed errors and misconceptions. Furthermore, interview transcripts were analyzed to provide additional context and support for the findings. The analysis was framed according to the preliminary genetic decomposition, detailed in the following section's theoretical framework.

Theoretical framework

The APOS framework explains how understanding of mathematical concepts develops through a hierarchy of mental constructions: action, process, object, and schema. Rooted in Piaget's work and constructivist theory, APOS theory focuses on modeling the mental activities students engage in when learning concepts such as applying determinants to solve systems of equations (Arnon et al., 2014)..

Genetic decomposition is one of the primary tools used in APOS research. It is a theoretical model outlining the mental constructions students are expected to form to grasp a mathematical concept (Arnon et al., 2014). Because it is initially a hypothesis, it is termed preliminary until it is supported by empirical evidence (Arnon et al., 2014).

APOS theory stresses the importance of learners' existing schemas in mathematics for building new knowledge. If earlier mathematical concepts are not fully encapsulated—meaning they cannot be applied flexibly to new problems—they may impede the learning of new ideas by remaining isolated from new concepts. Dubinsky (1997) pointed out that students' difficulties in linear algebra often arise from a lack of understanding of prerequisite mathematical concepts, which are not part of linear algebra per se but are essential to grasp it. This gap often results from students not having the chance to construct their own understanding of these concepts during earlier instruction. Supporting this view, Ndlovu and Brijlall (2015) highlight the significance of prior knowledge in acquiring linear algebra concepts, asserting that strong foundations in functions, equations, and algebraic reasoning contribute to developing schemas for systems of equations. Conversely, an absence of these mental structures hinders students' ability to understand solutions to systems of equations.

Preliminary Genetic decomposition on application of determinants in solving systems of equations

Drawing from the researcher's teaching and learning experience with matrix operations at secondary and tertiary education levels, the preliminary genetic decomposition was formulated. Moreover, the researcher's understanding of APOS theory significantly contributed to its development. Below, the specific constructions involving Cramer's rule and the inverse matrix method, which utilize determinants and matrix inverses respectively to solve systems of equations, are elaborated.

A model for the preliminary genetic decomposition of application of determinants in solving systems of equations

Prerequisite

The individual is expected to demonstrate action, process, and object conceptions of determinants and solutions to systems of equations, while exhibiting action and process conceptions of matrix inverses.

	Cramer's rule	Inverse matrix method
Action	The individual performs actions in a step-by-step manner, where each step serves as a prompt for the following one within the procedure. The individual is able to evaluate the determinant of an augmented matrix say A and the determinant of each matrix A_j the which is obtained from matrix A by replacing left column of A	Actions are executed by the individual in a stepwise fashion, with each step prompting the next throughout the procedure. The individual is able to calculate the determinant of matrix A for the equation Ax = b, followed by finding the cofactor matrix of A.,evaluate the adjoint matrix of A and

	by column vector $\bar{b}$. An individual can calculate the solution any systems of equations using Cramer's rule An individual can decide if Cramer's rule is possible by carrying out the steps.	then evaluate A^{-1}, using the formula: $A^{-1} = \frac{1}{\det A} adj\ A.$ The individual can calculate the solution of a any system of linear equation using inverse matrix method The individual can decide by inspection whether determinants and adjoint matrix can be found to decide whether the procedure is possible
Process	Actions of calculating solution of systems of equations using Cramer's rule are interiorised into processes. Can predict whether Cramer's rule is possible without having to do each step Explain how to use Cramer's rule	Actions of calculating solution of systems of equations using inverse matrix method are interiorised into processes The individual is able to determine in advance if the inverse matrix method can be used, without needing to complete all the steps. Explain how to use inverse matrix method
Object	Can explain conditions necessary for Cramer's rule to be applicable. The individual can recognize that the solutions obtained through different methods are equivalent. Can tell whether a system is consistent or inconsistent	Can explain conditions necessary for inverse matrix method to be applicable. Can see that the solution obtained by different methods are equivalent. Can tell whether a system is consistent or

	solution Individual can distinguish between Cramer’s rule and inverse matrix method.	inconsistent solution Individual can distinguish between Cramer’s rule and inverse matrix method

4 Results and discussions

Layer 1 Calculation of determinant

A concept is first seen as an action which requires an external prompt in order to carry out the action. When a student has worked with the concept, and can apply the transformation directly without going through all the steps, the concept is said to have been ‘interiorised into a process conception’ (Kazunga &Bansilal, 2018). A higher level of conception is the object level at which a person can carry out further actions or processes on the object which is now seen as a totality. Findings from the chapter indicated that some student teachers were limited to an action-level understanding of determinant calculation for square matrices, focusing on procedural steps. As in-service teachers advance, they move beyond this level, developing a conceptual understanding characterized by an interiorized process conception.

From content analysis and interviews of Question 1 some of student teachers seems to lack object conception of a singular matrix. Eight interviewed student teachers seem to be operating on action conception the inverse of a matrix because they fail to realise the a 2×2 matrix give during interview session was a singular matrix. Some participants like Mashie thought that any square matrix has determinant hence the inverse can be calculated. Though the participants taught determinant and inverse of a square matrix of order 2, some of them fail to get correct response to Q1.1 and 1.3. While some mistakes are what Siyepu (2013) terms slips, others reflect deeper misconceptions developed during high school mathematics. Some of the high school level mathematics is that a square matrix has determinant. When calculating the inverse of a 3×3 matrix some participants failed to calculate the adjoint matrix they simply transpose the give matrix.

Layer 2 Applying Matrix Determinants for Solutions to Systems of Equations

Cramer’s rule

Action conception

Findings from layer two content analysis and interviews reveal that certain student teachers remain at the action conception in solving systems of equations. Some did not even know how apply Cramer's rule correctly, they interchange the quotient and the divisor. Some prefer using other methods like Gaussian elimination method. The non-encapsulation of previously taught concepts for example determinants of matrices, adjoint matrix and integers was a barrier in acquiring extended mathematics concepts. Ndlovu and Brijlall (2015) observed comparable outcomes in their research, noting that mathematics undergraduate students' use of Cramer's rule to solve systems of equations was hindered by inadequate numeracy and algebra skills.

Process conception

From content analysis only 19 % student teachers manage to get correct responses on all question involving Cramer's rule. These participants seem to be reasoning at process conception. Further evidence from the interview where four out of 13 participants who were interviewed seem to be operating at process conception. These manage to explain when do you use Cramer's rule and inverse matrix method in solving system of equations correctly.

Inverse matrix method

Action conception

The data also revealed that most of the student teachers are at action conception. They had problems with the calculation of the adjoint matrix when solving systems of equations using the inverse matrix methods. Some of these participants simply transpose the given matrix or the augmented matrix and use it as the adjoint matrix. They then multiply the so called adjoint times reciprocal of determinant of the matrix multiplied by the original matrix was equal to identity matrix. The results might suggest that student teachers did not encourage their learners to check that product of so called adjoint times reciprocal of matrix determinant multiplied by original matrix result in the identity matrix. This might be the reason why the participants did the same thing.

Process conception

With application of determinants, there was evidence of some student teacherswhile a process understanding was achieved by some, the majority were still confined to an action conception. The participants at process conception manage to answer Q4 and Q6.1 correctly. Considering the notion of the inverse and the tasks completed, there are possibly 21 student teachers who are working at the process level or above. These participants manage to answer Q2, Q3, Q5 and Q 6.2 correctly.

It was quite exciting to note that some mathematics student teachers failed to write the formula connecting inverse of A, determinant of A and adjoint of A correct but manage to find the inverse of matrices in Question 1. It is possible to know the procedure of doing certain mathematical problems but on the other hand fail to write the correct formula.

Summary (Cramer's rule and inverse matrix method)

With respect to understanding use of Cramer's rule and inverse matrix method to find solutions for systems of linear equations, fifteen student teachers showed evidence of possibly having progressed toward an object conception, while most of the remaining student teachers demonstrated inconsistent performance across the questions, suggesting they were predominantly operating at the action conception.From content analysis nine student teachers manage to find correct solutions for all the questions requiring application of determinant. These student teachers seem to be at process conception. They were among the 25 % which were at process conception of determinants. Two out of eight participants who seemed to be at object conception in calculating determinant of square matrix manage to get correct response for all the questions requiring application of determinant.

Conclusions

Thechapter analysed the written responses and interviews from 116 mathematicsstudent teachers to assessment items based on the use of determinant of a matrix to solve systems of equations. It was discovered that participant carry out step by step calculating the determinant and applying it in solving systems of linear equations. The participants are struggling with concepts requiring deeper understanding of the core concepts. It is worth worrying about some participants who lack fluency in algebraic manipulation skillsand certain numeric operations. They lack in application on matrix algebraic concepts and application of determinant to solve systems of linear equations. The participants were taught during a short period of time and did not develop appropriate mental structures. They should be given more time through on line teaching

It was also found that students were generally able to complete items requiring procedural approaches, but encountered significant difficulty with tasks that demanded a deeper conceptual understanding. A notable concern is that some students demonstrated a lack of fluency in algebraic manipulation and certain numerical operations, which hindered their ability to meaningfully engage with matrix algebra concepts and their application to systems of equations.An additional issue lies in the limited time allocated within the curriculum for student teachers to develop the necessary mental structures. These structures are essential for understanding the underlying principles and relationships—particularly in the application of the determinant of a matrix. Structured opportunities that promote conceptual discovery are crucial.

Therefore, it is important for university administrators to design curriculum delivery plans that take into account these developmental needs. Such plans should provide student teachers with adequate time and support to engage deeply with mathematical concepts and develop the appropriate levels of understanding.

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Deep Science Publishing
https://doi.org/10.70593/978-93-49910-85-0

Chapter 4: A conceptual understanding of the quadratic function concept at undergraduate Level

1 Introduction

Mathematics is essential for the economic and social development of any country including Zimbabwe (Chauraya, 2023). Functions are one of the most crucial topics in mathematics as they are one of the first families of non-linear functions that students encounter. It is crucial for students to understand one of the components of functions which are quadratic functions. Despite being an important topic, students continue to struggle with conceptual understanding of functions (Simon et al, 2016).According to Simon et al. (2016), conceptual understanding reflects an integrated and functional grasp of mathematical ideas. Conceptual understanding enables the students to explain concepts in their own words, justify methods, and apply the knowledge different situations. This was supported by Jojo (2011) who was of the view that students with conceptual understanding are able to evaluate and correct their own thinking, and organize knowledge in a coherent way that supports further learning. According to Jojo (2011) if students have a conceptual understanding of a concept, they are more knowledgeable of the facts and methods. Their knowledge will be knowledgeorganisedinto a coherent manner making it possible for them to learn widely through connecting ideas to the knowledge they already have.

According to Bayazıt (2011) and Eraslan (2008), students have difficulty in understanding quadratic functions. The comprehension of as well as seeing graphs as tools for expressing the relationship between two variables is difficulty for students. Karim (2009) reported that students have difficulties while interpreting several quantities related to the quadratic functions such as extreme points, the leading coefficient and vertex and drawing the graphs. Many students struggle not only with solving quadratic problems but also with articulating and discussing their ideas, particularly in graphing and interpreting quadratic functions.Bayazıt (2011) highlighted students' difficulty in understanding the conceptual structure of quadratic functions, especially in graph interpretation. Students struggle with components such as the vertex, axis of symmetry, leading coefficient, and graph behaviour (Eraslan, 2008; Karim, 2009).Duarte (2010) observed that many students fail to grasp the abstract nature of quadratic functions, leading to persistent difficulties in advanced mathematics. Despite its foundational role in higher-level mathematics, the quadratic function remains one of the most challenging topics for undergraduate students. This chapter therefore, seeks to investigate the conceptual difficulties experienced by students when learning quadratic functions, particularly in relation to graphing and interpreting their features and propose a model for teaching quadratic functions.

1.1 Objectives of the Study

The objectives are to:

Determine the level of understanding demonstrated by students when drawing graphs of quadratic functions using the APOS (Action-Process-Object-Schema) theory.

Identify the challenges that students face in learning quadratic functions.

Propose a model for teaching quadratic functions.

1.2 Research Questions

The research questions guides this chapter;

What level of understanding do undergraduate students demonstrate when drawing graphs of quadratic functions, based on APOS theory?

What challenges do students face in learning the concept of quadratic functions?

What model of teaching quadratic functions can be developed?

2 Theoretical Framework: APOS Theory

Mathematics is viewed as an abstract subject, where many students struggle to develop procedural eloquence, whilst they lack comprehending the mathematical concepts needed to solve problems at hand or make connections between mathematical concepts (Makgakga, 2023). Various theories have been developed and used in order to find solutions to the challenges of teaching and learning mathematics One such theory is the

APOS (Action, Process, Object, and Schema) that has been used to understand the mental mechanisms that are related to mental development in mathematics. The theory was developed by Dubinsky (1991) with the aim of examining how learners develop an understanding of mathematics concepts. According to Dubinsky and McDonald (2001) APOS theory can be used in providing explanations of students' conceptual challenges as well as predicting failure or success in understanding mathematics concepts such functions, sequences and limits.

APOS Theory is used as a theoretical framework to analyze and explain students' cognitive development when learning quadratic functions, particularly in the context of graph interpretation and construction. The theory is used to in the identification of students' misconceptions and tailor teaching strategies accordingly. APOS Theory has been used in mathematics education because of its effectiveness in studying how students internalize and construct mathematical concepts (López et al., 2016). It provides insight into the progression of students' mental models and has been validated through various classroom studies. According to Bansilal et al. (2017) the theory is grounded in the idea that an individual develop mathematical understanding through mental and physical transformations. Although students may encounter the same mathematical tasks, the mental structures they construct can differ significantly, leading to varied approaches and problem-solving strategies.

APOS is an acronym for Action, Process, Object, and Schema (Arnon et al., 2014). These components represent different stages in the cognitive development of mathematical concepts. An action refers to the initial stage where learners respond to external stimuli by performing procedures step-by-step, often following specific rules or recalling facts from memory. At this level, learners are highly dependent on explicit instructions. Tziritas (2011) identifies the following as indicators of learners operating at the action level:

- Viewing a function as a relationship between two sets.
- Substituting values into a function and calculating outputs.
- Approaching problems sequentially, one step at a time.
- Recalling definitions or procedures verbatim without deeper understanding.

Process is when a learner begins to reflect on and internalize actions, these actions evolve into mental processes. At this level, learners can perform operations mentally and perceive procedures as whole entities. Chimhande (2017) describes learners at the process level as those who:

- See a function as an operation or transformation that processes inputs into outputs.

- Understand the overall purpose of a function without relying on step-by-step computation.
- Provide definitions that reflect an integrated view of input–process–output systems.
- Visualize problem-solving holistically rather than focusing on isolated steps.

Object is when a learner encapsulates a process into a mental construct that can be manipulated as a whole, it becomes a cognitive object. This stage allows for abstraction and the ability to reflect on functions as complete entities. According to Dubinsky and McDonald (2002), learners at the object level:

- Recognize a function as a mathematical object or noun.
- Perform operations that act upon functions themselves, such as transforming or combining them.
- Demonstrate flexibility in reasoning about functions as manipulable entities.

Schema is the highest level in the APOS model that refers to the integration of multiple actions, processes, and objects into a coherent mental structure. Schemas are dynamic, evolving as learners connect new knowledge with existing conceptual frameworks. Brijlall and Maharaj (2008) defines a schema as a structured network of related ideas that learners use to understand and solve mathematical problems. Kazunga and Bansilal (2020) emphasize that schema formation is crucial for long-term retention and the transfer of learning across different mathematical contexts. Students with well-developed schemas:

- Use formal definitions of functions to generate or analyze examples and non-examples.
- Transition smoothly between different representations, such as equations and graphs.
- Understand the relationships between key features of functions (e.g., critical points) and their graphical representations.
- Integrate new concepts with prior knowledge, enabling them to apply known strategies in unfamiliar situations.

An important tool used in APOS-based research is genetic decomposition, which involves hypothesizing the mental structures a student must build to understand a specific mathematical concept. This tool helps educators anticipate learning difficulties and design instructional sequences that support conceptual growth. APOS theory offers a robust lens through which to examine the conceptual development of students learning quadratic functions. It enables the identification of learners' cognitive levels and informs targeted interventions to facilitate meaningful understanding.

Quadratic functions

In engineering and science, quadratic functions are utilized to obtain parameters of different kinds. This is a polynomial function that is defined in all numbers. According to Parent (2015), a quadratic function can be expressed as an equation of $f(x) = ax^2 + bx + c$., where a ≠ 0.Two is the highest power of a given variable in vertical (quadratic) functions. A parabola is a graph of quadratic functions; these objects may open upward or downward, and their width or height may vary, but they all have the same basic "U" shape. The direction of the curve is dependent on which degree corresponds to the coefficient of highest degree.

Those functions that can be described qualitatively and have variable rates of change are classified as quadratic functions (Strickland, 2011). Students can use quadratic functions with output values that are representative of both x and they axes, making it easy to understand the rate of change as its gradient of the tangent at point without worrying about scaling. Learning about the significance of adjusting the y-axis' scaling when reasoning graphically about change/gradients is achievable through quadratic functions by examining the differences in graph visual appearance caused by changing scaling.

Whether the translation was horizontal or vertical is visually unclear when translating linear functions. .Students may wonder why focus on the y intercept in the $y = mx + c$ representation, especially as the $ax + by = c$ representation gives both intercepts equal importance. The explanation of quadratic functions, as proposed by Good and Lavigne (2018), involves understanding the direction of translation and presenting a new crucial interpretation of intercepts on the x-axis. This helps to clarify the problem. It is important for students to consider both qualitative traits and also rates of change, constants by y intercepts and zeros. Only in quadratics can students use algebraic and arithmetical manipulation to demonstrate the connections between input/output values, different algebra representations, and various graphical representation(s) (Berger und al. 2020). The quantitative characteristics of phenomena can be posed using quadratics as a foundation for various inquiries. Including inquiries about growth and decline, change rates (uplifts and decreases), or attaining specific values like zero and the location of maxima and sub-maximates. Students in higher mathematics can use imaginary numbers to solve problems and question quadratics by utilizing the necessary expression i - the square root of -1.

Student understanding of quadratic function.

For students to understand quadratic equations they must possess the ability to graph and predict the effects of each coefficient in order to comprehend quadratic functions. They are second-degree polynomial functions of the type $ax^2 + bx + c$in which a, b and c are constants and a≠0. Quadratic function can be represented by a graph or through algebraic expression of quadratic function is called a parabola. F denotes a quadratic function, with x being the independent variable, $f(x) = ax^2 + bx + c$. Makgakga (2023) noted that there are the three forms a quadratic equation which are Standard form which can be written in the form $y= ax^2 + bx + c$., Factored form which can be written in the form $y = (ax + c)(bx + d)$ and Vertex form which can be written in the form $y = a(x + b)^2+ c$ where a, b, c are constants or coefficients of variables.

How to draw graphs of a parabola

The graph will open downwards if a is negative and upwards when it is positive. If the quadratic coefficient is negative, then both ends of the parabola point downwards.

The effects of varying *a*

There is a further transformation that results in stretching arms of parabolas producing a new parabola that is not congruent to the original one. If the value of *a* becomes bigger, the graph become thinner and if value of a becomes smaller, the graph becomes wider. This was supported byMakgakga (2023) reported that the coefficient of the quadratic term *a* determines how wide or narrow the graph is. If a is negative the graph face down ward.

Effects of varying *b*

Consider x^2- or + b*x* then the shape does not change because it's a quadratic function. The graph shift down to the left when the graph is positive and down to the right when the graph is negative.According to Makgakga (2023), changing the value of *b* when positive it shifted down to the left and when it is negative it shifted down to the left.

Effect of varying *c*

Makgakga (2023) noted that a change in the value of *"c"* will move the vertex of the parabola down or up and "*c"* is all the time the value of the y-intercept (down if c is negative and up if c is positive).

Determining vertex on the graph

When plotting parabolas, special points in the graph are included. The *y*-intercept is a point where the graph intersects the y-axis and *x*-intercepts is the points where the graph intersects the *x*-axis. A vertex is the point that defines maximum or the minimum of the graph. The vertical line through the vertex is the line of symmetry (also known as the axis of symmetry).

3. Challenges faced by students when learning quadratic functions

The learning difficulty faced by students in quadratic equations is as a result of poor teaching methods (Eraslan,2008). According to Makonye (2011), errors and misconceptions may also result from gaps in algebra such as removing brackets and factorisation pose a challenge in the solving of quadratic functions. This was supported by Díaz and Poblete (2018) who said one area of persistent difficulty is algebra, which is correlated to functions, in that they model a dependent relationship between one quantity and another. Eraslan (2008) remarked that students do not have the capability to make and investigate mathematical connections between algebraic and graphical aspects of the quadratic functions. If learners fail to correctly factorise a given quadratic functions they would end up having wrong critical values that would in turn result in a wrong solution.

Mathematics teachers' comprehending of quadratic functions is crucial for the success of students as it seems to be an agreement that, for most students, solving and understanding quadratic functions is difficult because of the need to make associations between various representations of the function, and the connections between the a variety of ways in which the quadratic equation can be articulated (Didis et al., 2011).

Makgakga (2023) stated that the most challenging task for students is to grasp the function's definition and its relationship to geometry. According to different studies, the understanding of images and pre-images in both algebraic and graphical forms is only partially established (Brijlall & Maharaj, 2008). Makgakga (2023) reiterates that students find it challenging and perplexing to draw graphs of quadratic functions, even though graphing functions is an essential aspect of learning quadripoly. In (2008), Eraslan emphasized the need for students to be familiar with interrelated concepts such as turning points, intercepts, and the impact of quadratic function parameters. This was outlined in his work.

The comprehension of quadratic functions is often hindered by various obstacles encountered by learners. There are both conceptual and procedural issues. The majority of erroneous ideas are created by students who have procedural knowledge (Siyepu, 2013). Misconceptions are erroneous and frequently employed ideas (Parent, 2015). Makgakga's (2023) suggested that students bring different understanding of the coefficients, a, and b, in the quadratic function when teaching in classrooms. It is uncertain for some students if the coefficients will affect the vertex. In his work, Makgakga (2023) observed that a student who is academically proficient may seem to comprehend quadratic functions, but their understanding of the concept itself is not always straightforward. The development of misconceptions can be attributed to either over-generalizing an essential, correct idea or interference from everyday knowledge (Parent, 2015). Learning to differentiate between functions and non-functions is challenging for students, who may also be unable to use appropriate notation in the graph of a function.

Brijlall and Maharaj (2008) assert that graphing quadratic functions is a crucial aspect of the subject, but students find it challenging and perplexing to depict them visually. Problems are a result of students having to remember effects of a, b and c (negative and positive) when given the variables a and b. Students must possess a thorough understanding of the effects of a, b and c, as well as the ability to draw Graphs (Brijlall and Maharaj, 2008). Teaching quadratic functions requires the ability to grasp related concepts like turning points, intercepts, and effects of quadratic functions, as per Eraslan (2008).

4. Learning and teaching strategies of mathematics

According to Djamarah (2010), the teaching approach is a method used to teach and achieve the desired outcomes. Teachers employ teaching methods that correspond with the characteristics of students they encounter to teach subjects to them, including demonstrations, discussions, laboratories, projects, contests using tangible objects and supervised experience. A teacher needs to possess conceptual comprehension and procedural comprehension of a given concept in order to create teaching methods that promote understanding (Makgakga, 2023). Therefore, a teacher should be capable of adapting the learning process to suit the needs of students. Mathematics can only be learnt through doing things, making things, noticing things, arranging things and then reason about things. There should be a culture that should be developed among children for them to create interest in the subject as most students would be interested in subjects like music and art. Makgakga (2023) reported that discussing, refining, sharing and questioning are acts taken by human beings either individually or together in exchange of ideas. This clearly shows that through questions misconceptions and grey area are cleared and students will have a clear understanding of the concept under discussion. This method should also be made use of to ensure the complete understanding of students of all concepts in quadratic functions. This was supported by Makgakga (2023) reported that students understand and retain knowledge longer if they discover it for themselves. Learning by discovery involves active participation by learners.

Teachers' use of specific teaching methods is believed to be more effective in achieving the learning objective, as stated by Qudsyi et al. (2011). Learning style is influenced by learners, objectives, situational contexts, facilities, and teachers. The teacher will aim to teach with precision by using an accurate method. Thus, instructors should utilize techniques that can support teaching and learning activities to make it an efficient means of achieving the objective of teaching. Hence, teachers utilize learning methods to attain the objectives in teaching and learning.

A positive classroom environment is crucial to the learning process. A comfortable, peaceful, and enjoyable environment conducive to learning is desirable. Students' brain function is influenced by their interests or mood, so if they are comfortable and happy while teaching and learning, the brain will readily accept identified material. Students who experience discomfort and unproductive behavior in class may seek an immediate end to the learning process. This can lead to boredom and laziness while listening to teachers. They are not concentrating on the lessons taught by their teacher, leading to distraction. Why? Through a variety of activities, the student works hard to achieve learning achievement in the process of teaching and learning. According to Zuldafrial (2011), a good achievement is one that can be attained by students through serious and hard work, which leads to satisfactory achievement. The learning atmosphere plays a crucial role in enhancing student performance. A crowded or noisy learning environment can cause disturbance to other students who are engaged in learning activities. In addition to the raucous behavior of classmates, the learning environment in class is influenced by teacher-student interaction, building ventilation, room lighting, wall hangings and wall state.

Having a positive attitude towards the subject and teachers is crucial for students to comprehend their lower level education in secondary schools. According to Akinsola & Olowojaiye (2008), the attitudes of students are likely to either stimulate or discourage further mathematics education. According to Anthony & Walshaw (2007), the learning process is heavily influenced by attitudes. It is the responsibility of educators and all those involved in education to aid students in developing a positive outlook on mathematics.

Students' learning is influenced by the attitude of teachers who teach them in the classroom. Mathematicians have a significant influence on their learning approach, with teachers having varying attitudes and beliefs towards mathematics (Makgakga, 2023). To achieve this change, it is necessary to shift from an analytical-based teaching approach to focusing on problem-solving, which takes into account the teacher's attitudes. Beliefs, emotions, social context and content knowledge are the main influences on attitudes and practices in mathematics teaching.

5 Methods and materials

A qualitative research design that includes any information that can be captured that is not numerical in nature was used in this study. Qualitative research methodology enables one to use diverse research strategies to collect data. It enables for the voice of the participants to be heard. The study intends to gain concrete, contextual, in-depth knowledge about conceptual understanding of quadratic functions, so a case study design was used to inestigate students' understanding of the quadratic function concepts. A case study is characterized by gathering data for a long period. Interviews, questionnaires and document analysis were used to collect data.

The group consisted of 20 trainee teachers who were studying for a mathematics education diploma at s Zimbabwean universities. ". 10 trainee teachers were chosen through a random selection process. When a sample is founded on the knowledge of sex and the purpose of research, it is called essentially purposive or judgmental. Through the use of judgmental sampling, the researcher chose a sample of interviewees to determine the aspects of the population that make this study valuable and easy to handle. Participants were chosen for the purposive sampling method based on their knowledge or interactions related to the research question.

An interview was conducted in an interactive environment where two or more individuals participate in a conversation that is initiated and coordinated by the interviewer to gather information specific to whichever area of interest they are interested in. Written answers were analyzed by trainee teachers to ensure clarity and explanations. Semi-structured interviews were utilized in this study to obtain additional information on the students' written work. The effectiveness of qualitative research is largely due to the use of interviews, which aid in comprehending and understanding the opinions, feelings, behavior, and experiences of research subjects.

The conceptual comprehension of quadratic functions was assessed using a test. The test was used to determine whether the students had fully grasped the concepts of quadratic functions during teaching and learning was one of the objectives of this test. The instruments were tested by a math teacher to guarantee their validity. To verify the instruments' reliability and validity, a pilot test was conducted.

6 Results and discussions

Pseudo names (A to J) were used for the trainee teachers. The questions and answers provided by the trainee teachers are provided in this section.

Table 4.1 Question1 (What do you understand by the word quadratic functions)

Category	**Action**	**Process**	**Object**	**Schema**
Indicator	Equation whose highest power of the unknown is 2	A relationship between range and domainwhose highest power of the unknown is 2	A function whose highest power of the unknown is 2	Is an algebraic expression whose highest power of the unknown is 2 and its graph is parabolic.
	2	4	3	1

Analysing these results in table 1 shows that trainee teachers lacked a clear understanding of the meaning of the word quadratic functions. Most of the trainee teachers define the word quadratic equation instead of quadratic functions meaning trainee teachers where confusing the two words. These findings described a big gap between proposed definition and the trainee teachers responds. Some of the trainee teachers just concentrated on the quadratic aspect only and ignored the term functions. However, all the trainee teachers remember that the highest power of quadratic function is 2.Some of the written work extracts and some for selected trainee teachers are shown below;

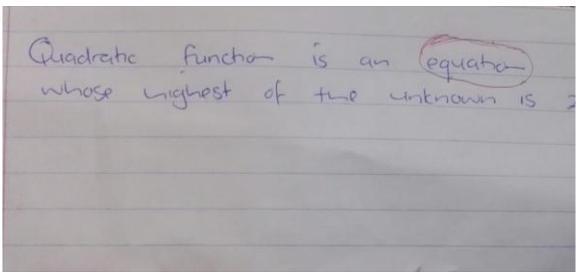

Fig 4.1: trainee teacher E’s understanding on question 1

From this question of defining, it clearly shows that trainee teachers were only able to define the word function, from fig 4.1, it is clearly showing that trainee teacher E just assumed that equation with two as the highest power of the unknown is quadratic function. This was supported by Didis et al (2011), who said understanding quadratic functions can be conceptually challenging because of the need to make connections between various representations of the function, as well as the connections between the various ways in which the quadratic equation can be expressed. This shows that the trainee teacher is at action stage in APOS stage where trainee teachers uses the term without a clear conceptual understanding of the word. Below is the correct definition of quadratic function;

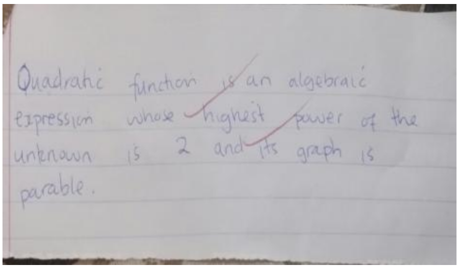

Fig 4.2: Trainee teacher H correct answer

On fig 4.2 above, it clearly shows that the trainee teacher has master the concept of quadratic function. This shows that the trainee teacher is at schema level, where the trainee teacher was able to use logical definition to determine whether a given relation is a function or a non-function.

Trainee teacher E's interview response on question 1

Researcher: Is quadratic function and quadratic equation the same?

Trainee teacher E: Yes

Researcher: Why did you say so?

Trainee teacher E: According to what I have observed, both have highest power of the unknown is 2 so they are same.

Researcher: So how do you write the expression of the quadratic function

Trainee teacher E: $ax^2+bx+c=0$, *where* $a \neq 0$

Researcher: Ok, so why did you not put $f(x)$

Trainee teacher E: As I mention before they are same, so you can use both.

From the above interview, it clearly shows that the trainee teacher confuses the quadratic equation and quadratic function. The trainee teacher was unable to differentiate between these words as long as they contain two as highest power of the unknown. This interview shows that the trainee teacher is at action level where the trainee teachers is unable to differentiate the difference between quadratic function and quadratic equations because both have the highest power of the unknown 2.

Table 4.2 Question 2 **(**Describe how your draw the graph of $f(x) = x^2$-2

Category	**Action**	**Process**	**Object**	**Schema**
Indicator	Shapes of the graph	Knows how to draw but not explaining	Mentioning of points	Action, process and object
Number of responses	2	1	5	2

Analysing these results in table 2, it is evident that the trainee teachers lacked a clear understanding of the explanation on how to draw the graph of quadratic function and their level of cognitive development. Some trainee teachers fail to explain the terminology used when drawing the graph, it implies that they were not in a position to draw the graph correctly. One of the trainee teachers fails to explain whether the graph is in which shape (open upward and open downward) and this means the trainee teachers did not understand the concept at all. These results were supported by Makgakga (2023) who reported that trainee teachers leave the part of explaining whether it open downward or open upward when drawing quadratic function graph. Some of the written work extracts and some for selected trainee teachers are shown below.

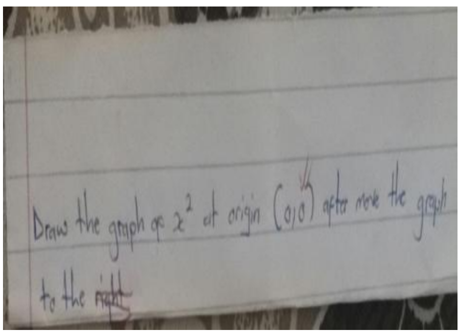

Figure 4.3: trainee teacher D's understanding on question 2

Out of 10 trainee teachers who were selected, only 4 trainee teachers answer the question correctly. Some trainee teachers did not use correct terminology and some did not indicate whether the graph open upward or downward. The trainee teachers leaves important like the shape of the graph and also the vertex of the point.The other trainee teacher did not understand the effects -2 instead of move two steps downward along the y axis, the trainee teacher moves two steps to the left. Meel (2003)'s work shows that the use of diagrams facilitates the explanations of the transformation of the graph. From figure 4.3 above, it shows the trainee teacher D's work. The trainee teacher has an idea on how to draw the graph of $f(x) = x^2$. The trainee teacher shift the graph two steps to the right along the x-axis. This shows that the trainee teacher confuses the two quadratic function which are $f(x) = x^2$-2 and $f(x) = (x - 2)^2$. Below is the answer on what the trainee teacher was supposed the do;

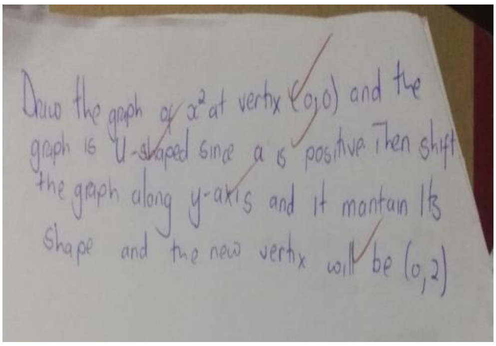

Figure 4.4: trainee teacher G' correct answer

The above fig 4.4 shows the correct answer of trainee teacher G and the answer contains all important information to draw the graph. It shows that the trainee teacher had understood the concept of quadratic functions and how to draw it. The trainee teachers write all the vertex points and this show that the trainee teacher was at schema level in cognitive development. This was supported by Ibeawuchi (2010) who argued that changing the value of "c" will move the vertex of the parabola up or down and "c" is always the value of the y-intercept (up if c is positive and down if c is negative).

Trainee teacher D's interview response on question*2*

Researcher: What the sharp of the graph you explained?

Trainee teacher D: The graph is U sharped since the coefficient of x^2 is positive.

Researcher: So why did you not mention it in your exercise.

Trainee teachers D: I did not think it was necessary since we know the graphs of a parabola.

Researcher: Ok! What about the effects of -2 on your graph.

Trainee teacher D: The graph moves to the right by two units.

Researcher: Is that not the graph of $(x-2)^2$.

Trainee teacher D:Ummmm, no that graph you mention will move along y-axis by 2 units going upwards.

From the above interview it shows that trainee teacher D leaves an important information and it was marked wrong. The trainee teachers have forgot to mention the new points in which the graph had moved to and also fail to explain why they have described the way they did. Thistrainee teacher's answer lacks some explanation and seemsto operate on action and object level on APOS theory as the teacher was able to draw the graph of quadratic functions but did not clearly explain the process.

The researcher went on interview the trainee teachers to get the clarity on why the trainee teachers fail to interpret the effect of -2 in the function which says x^2-2. It looks like the trainee teachers have fail to interpret correctly the effects of -2 and this led to more than half of trainee teachers who fails the question. The trainee teacher was explaining the graph of $f(x) = (x-2)^2$ instead of $f(x) = x^2$-2.

Table 4.3 Question 3(Draw a graph of $f(x) = -x^2-3$

Category	**Action**	**Process**	**Object**	**Schema**
Indicator	Graphing using table of values	Graphing without table of values but fail the effects of c	Transformation of varying a, b, and c.	Action, object and object
Number of	1	3	2	4

responses				

The question requires the responded to draw a graph of $-x^2$ which face downwards and moves by 3 downwards. One trainee teacher was an action level where the traineeteachers did not put effect of negative *a*, 3 trainee teachers were in a process sincethey forget the effects 3, 2 trainee teachers at object since they draw graph without points and 4 trainee teachers are in schema level where trainee teachers have done all stages correctly. The written work of some of the trainee teachers are shown below;

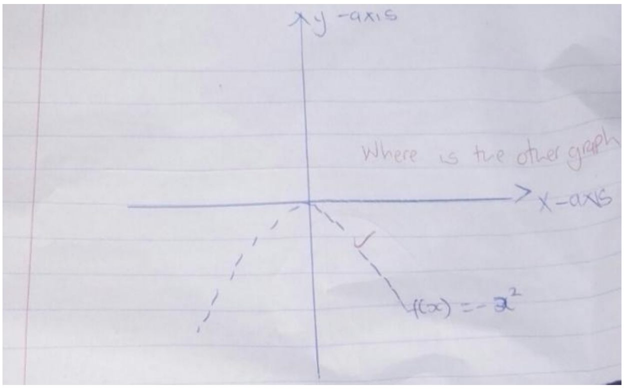

Figure 4.5 trainee teacher C's understanding on question 3

From the above, fig 4.5 the trainee teacher draws the graph of $-x^2$ and did not draw the graph of $-x^2-3$. The trainee teachers lack the concept of drawing the quadratic function graph. This shows the trainee teacher is at action level were where the trainee teacher fails to memorise what she was taught. Below is the correct answer of the graph;

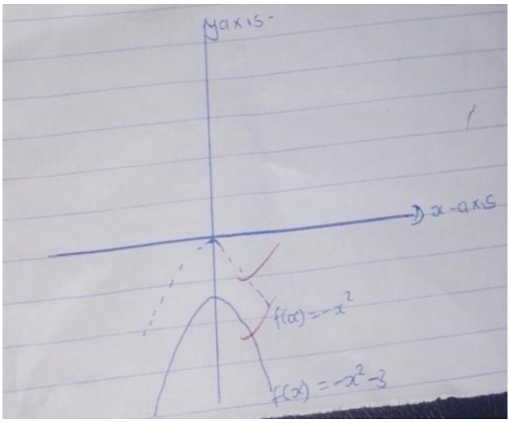

Fig 4.6 trainee teacher F correct answer

The above figure 4.6 shows the correct parabola and all the effect of a>0 which the parabola graph will face downwards and effect of b where the graph moves by 3 to downwards. This shows that the trainee teachers have master the concept and all the procedure are known.

Trainee teacher C's response on interview

Researcher: Why didn't you draw the effect of -3

Trainee teacher C: I had forgotten how to shift the graph

Researcher: So now you know

Trainee teacher C: I'm not sure of what is in my heard

Researcher: Ok, can you explain how you would draw

Trainee teacher C: I move or shift the graph 3 steps downwards

From the above interview it shows that the trainee teacher did not have confidence and end up not drawing the graph. This trainee teacher is on action level where the trainee teachers is not sure on what to draw and unable to construct but have the procedure in the heard. According to Tziritas in chapter 2 action is when the trainee teacher can recall a fact from memory and fail to construct it on the ground.

Table 4.4 Question 4, Draw the graph of $f(x) = (x + 5)^2$

Category	**Action**	**Process**	**Object**	**Schema**
Indicator	Graphing using table of values	Graphing without table of values but fail the effects of c	Transformation of varying a, b, and c.	Action, object and object
Number of responses	1	3	2	4

The question requires the trainee teachers to draw a parabola in the form $a(x+b)^2$ where a and b are constant. Most of the trainee teachers where familiar with the effects of *a* only. The trainee teachers focused on changes which occur when the value of *a* takes positive or negative. For *b*, most of the trainee teachers have forget the procedural knowledge. Below is trainee teachers' F written work,

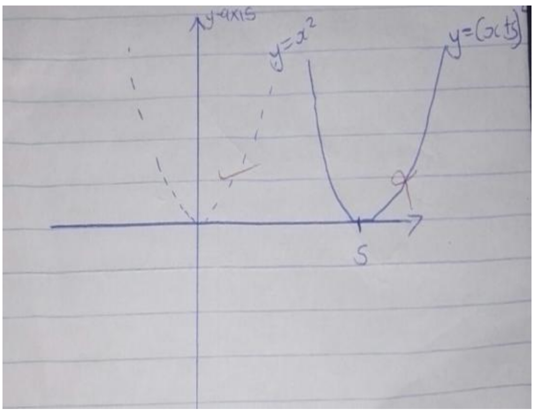

Figure 4.7 trainee teacher F understanding on question 5

From above it shows that the trainee teacher was familiar with the graph of x^2. The trainee teacher had no idea on the effects of b. The trainee teacher instead of shifting the left, the trainee teachers shift to the right. This show that the trainee teacher did not understand concept of shifting the graph thus conceptual understanding was lacked and this shows that trainee teacher was in process level. Below is the correct graph;

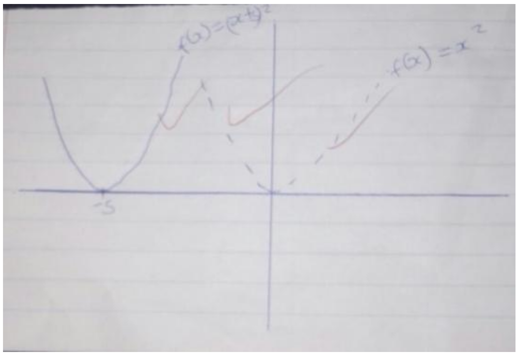

Figure 4.8 trainee teacher A’s correct answer

The above shows the correct graph with effect of *a* and *b*. The graph shifts 5 units to the left since b>0. This shows that the trainee teacher had understood both the procedural and conceptual knowledge followed when drawing a graph.

Trainee teacher interview response question 4

Researcher: How did you come up with your graph

Trainee teacher: I first draw the graph of x^2 *which is a parabola which faces upwards and shift the graph by 5 steps to right maintaining its shape.*

Researcher: Ok, so why did you move to right not left

Trainee teacher: Because 5 is positive so it moves to the right if it was negative it was going to move to the left.

The trainee teachers a graph of $(x+b)^2$ and of $(x-b)^2$ and the trainee teachers was confidence. This show the trainee teacher operate on the process stage where the trainee teacher has the knowledge have the knowledge but fails to comprehend it.

Table 4.5 Question 5 (Stretch the graph of $f(x) = x^2 - 2x + 1$, showing clearly the effects of each coefficient)

Category	Action	Process	Object	Schema
Indicator	Shape of the graph	Points to fulfil the graph	Effects of a and c	Correct answers
Number of responses	0	3	5	2

The question required the learner to draw the graph in the form ax^2+bx+c, where $a \neq 0$. Some of the trainee teachers were familiar with the effects of varying a and b only and most of the trainee teachers were confused on how to draw that kind of function. These findings were similar to Makgakga (2023) who indicated that the value b was always a problem to the trainee teachers as it tends to ignore the sign of value of b and also that the value of b changes the vertex of the parabola. Some of the written work extracts and some for selected trainee teachers are shown below;

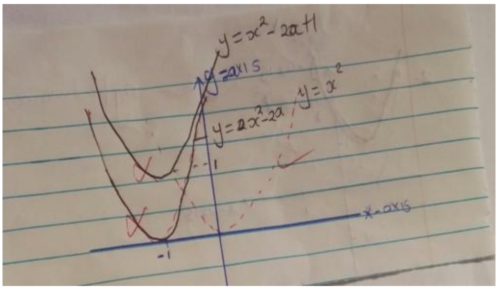

Figure 4.9: trainee teachers A's understanding on question 5

From this question, it shows that the trainee teachers have not understand the concept of drawing graphs especially on how b shift if it negative or positive. Out of ten trainee teachers who right the test only 2 trainee teachers know how to draw the graph and 5 of them did not know how -2 of the function x^2-2x+1. Most of the trainee teachers shifts the graph to the left because they thought it goes to the negative side and some of the actually knows that the graph moves to the right but leaves the concept of shifting the graph downward. Trainee teacher I's work shown in the diagram above indicate that the trainee teacher has an idea of drawing this graph but leaves out the concept of shifting it down. This was supported by Makgakga (2023) who said that the trainee teachers forget the concept of shifting the graph downwards to the right or left. Few trainee teachers did not even draw the effects of, which is, if c is positive it shifts upwards and vice versa. The correct is shown below;

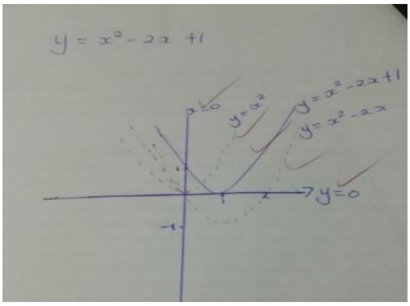

Figure 4.10: trainee teacher J correct answer

The above figure 4.10 shows the correct graph of $f(x) = x^2 - 2x + 1$. This graph shows the both the shifting of b and of c. Although the trainee teacher fails to draw the smooth curve but it shows that, the trainee teacher had understood the concept of drawing the graph. The trainee teacher had the conceptual understanding of the effects of a, b and c andhow to shift the graph.

Trainee teacher A's response on question 5

Researcher: How did you draw the graph of f(x)= x^2 *-2x+1*

Trainee teachers A: I first draw the graph of x^2 *which is a parabola facing upwards because it is positive then shift the graph to the left by one unit and finally shift the graph upward to the y-axis by one unit.*

Researcher: Why did you move b by one unit to the left

Trainee teacher A: Because I learnt that the effect of b will shift by half and I move to the left because it is negative.

Researcher: Ok what about the concept of shifting downward to the left or downwards to the right.

Trainee teacher A: Ummmm, I don't remember where it can be used.

From the above interview, the researcher note that trainee teachers face problems in trying to draw a graph on quadratic function. Trainee teacher A said that it was not an easy question because it requires high order thinking. According to the interview the trainee teacher faces the problem of shifting b. This interview shows that the trainee teachers did not understand the concept of shifting the graphs. According to the APOS theory the learner is at process stage when an individual examines and reflects on an action or series of actions, that action can be internalized into a mental process.When working with standard form, interpreting the *c* parameter (the *y*-intercept) appears to be more straightforward for most trainee teachers. Trainee teachers seem to have a partial understanding of the *a* parameter, though the differing roles of the *a* parameter in the standard and the vertex forms may be a point of confusion. When working with the vertex form, most trainee teachers can readily identify the vertex, but many still have difficulty with transforming the graph, even when using this form.

Table 4.6 Question 6 (Draw the graph of $f(x) = (x-4)^2-5$ and state whether it is a minimum or a maximum and at which point)

Category	**Action**	**Process**	**Object**	**Schema**
Indicator	Shape of the graph	Points to fulfil the graph	State whether it is minimum or maximum	Correct answer
Number of responses	2	4	4	0

The question required the learners to draw the graph from $y = a(x + b)^2 + c$ where a, b, c are constants or coefficients of variables and (b,c) is the vertex. Most of the trainee teachers were not familiar with this form which is the vertex form. The trainee teachers did not know how to draw that kind of the graph and how to state whether it is a minimum or maximum. Some of the trainee teachers only identify the point and fails to identify whether it is minimum or maximum. These trainee teachers thought that the graph that faces upwards if is minimum and those that faces the downward is maximum. Some of the written work extracts and some for selected trainee teachers are shown below;

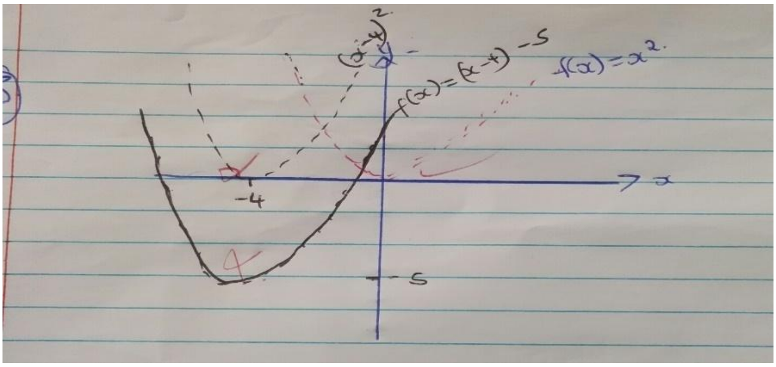

Figure 4.11: trainee teacher G's understanding on question 6

Out of 10 trainee teachers who wrote the test only no trainee teacher has done it well both drawing and able to see that if it is a minimum or maximum. Trainee teacher G marked wrongly because, she or he fails to draw a smooth curve which will pass through x-axis at (0, 2) and (0 ,6). Makgakga (2023) reported that many trainee teachers fail to draw a smooth curve and this may actually lead to wrong coordinates on x-axis and y axis. Trainee teacher B have failed to draw the graph of $f(x) = (x - 4)^2$, instead of shifting the graph to the right, the trainee teacher shifts the graph 4 steps to the left. The trainee teachers thought that the negative sign on the function means that it moves to the left. According to Makgakga (2023) said that trainee teachers face challenges graph with $(x + b)^2$, they interpret wrongly. On fig 4.1 above, it shows the work of a trainee teachers treated -5 how she or he treated -4. The trainee teacher ends up shifting five steps downward, she or he end up shifting five steps upwards. Trainee teacher B also did not know whether it was a minimum or a maximum point and at what point. The correct answer is shown below;

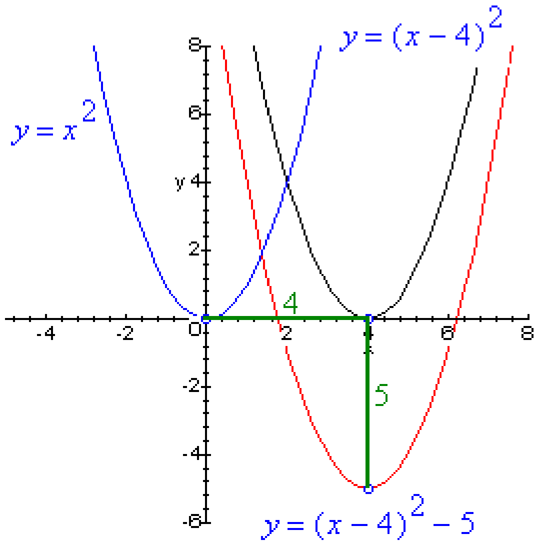

Figure 4.12: correct answer

The above fig 4.12 shows the correct answer also that it is a minimum point at vertex at (4, -5). The above graph shows how was trainee teachers supposed to draw and label the vertex. In order for trainee teachers to draw the correct graph they were supposed to first draw the graph of x^2 then it was supposed to be followed by moving 4 steps to the right since it was negative and then move 5 steps downwards because of a negative value. The trainee teacher would have been move 5 steps downwards before moving 4 steps to the right it was also correct. Also, the important thing was the graph pass through 2 and 6 at x-axis.

Trainee teacher B's response on question 6

Researcher: How did you come up with your graph

Trainee teacher B: I first draw the graph of x^2 which is a parabola which faces upwards and shift the graph by 4 steps to the left maintaining its shape.

Researcher: Ok, so why did you move to left not right

Trainee teacher B: Because 4 is negative so it moves to the left if it was positive it was going to move to the right.

Researcher: Ok what about the effect of -5

Trainee teacher B: I move the graph down ward maintaining its shape

The trainee teacher had an idea on the knowledge but fail to identify them. The trainee teacher has no idea on shifting the negative sign of value *h* but on the shifting of *k* the trainee teacher B had no problem. This mathematics requires high order thinking and most of the trainee teachers did not understand what the question requires. The research conclude that most trainee teachers do not enjoy graph work, which a weakness for most Mathematics trainee teachers. This was supported by Vaiyavutjamai and Clements (2005) noted that when working with the vertex form $y = a(x - h)^2 + k$, a trainee teacher attended to the sign of the *a* parameter but not to its value.

Proposed Model for teaching quadratic functions

A research-informed model for teaching quadratic functions, the Q.U.A.D. Model, aims to address conceptual misunderstandings, support diverse learners, incorporate student-centered strategies, and foster positive attitudes toward math. Q.U.A.D. stands for: Question and Elicit Prior Knowledge, Use Multiple Representations, Apply Collaborative & Discovery-Based Learning, and Diagnose & Differentiate Instruction. Figure 4.13 shows the Q.U.A.D. Model.

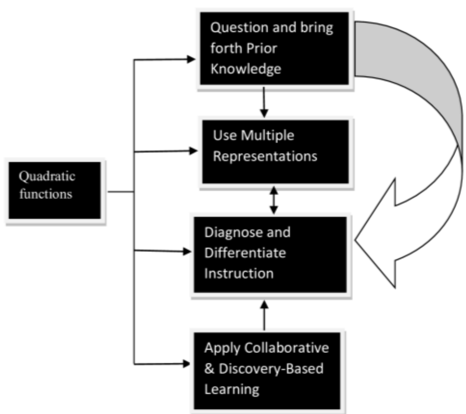

Figure 4.13 Q.U.A.D. Model

The Q.U.A.D. Model is a cyclical, learner-centric framework aimed at improving students' comprehension of quadratic functions by integrating constructivist and direct instructional strategies. It starts with the "Question" phase, where educators draw out students' prior knowledge and identify common misconceptions about quadratic expressions, equations, and functions. This initial assessment is crucial for recognizing students' basic understanding and sets the stage for more in-depth involvement. The subsequent phase, "Use," focuses on various representations by connecting algebraic forms (like standard, factored, and vertex forms) to their graphical representations and real-world contexts. Teachers facilitate learners' exploration of how parameters such as a, b, and c influence the shape, orientation, and location of the parabola, thereby reinforcing conceptual understanding through both visual and symbolic reasoning.

The "Apply" phase promotes collaborative, inquiry-driven learning, where students collaborate in pairs or groups to create graphs, evaluate transformations, and share their reasoning with each other. This stage fosters discovery learning, which has been proven to enhance long-term retention and deeper comprehension. Lastly, the "Diagnose" phase entails ongoing formative assessment and tailored instruction. Educators utilize students' responses and graphing activities to modify teaching approaches, provide extra support for learners who need it, and challenge advanced students. By moving through these stages, the Q.U.A.D. Model addresses a variety of learning needs, fosters active engagement, and establishes a more robust conceptual framework for understanding quadratic functions.

The model emphasizes understanding prior knowledge, employing various representations, and utilizing tools like Desmos for visualizing transformations, ultimately enhancing learners' grasp of quadratic functions and their applications.

Students often find it challenging to interpret parameters like b and differentiating shifts, so visual tools and comparisons enhance understanding. The Q.U.A.D. model promotes collaborative learning, allowing students to engage in group tasks, match through content can lead to misconceptions. The model fosters an inclusive environment, blending teacher guidance with student discovery for effective quadratic instruction.

Conclusion

This chapter has examined the obstacles that learners encounter when trying to comprehend and graph quadratic functions, along with the teaching strategies that mathematics educators use to overcome these challenges. The findings indicated that numerous students find it difficult to understand the impact of parameters in various forms of quadratic equations. The dependence on a singular teaching demonstration strategy has led to diminished learner engagement and performance. In light of this, the proposed Q.U.A.D. Model introduces a flexible and inclusive framework that incorporates diagnostic teaching, multiple representations, inquiry-based learning, and continuous assessment. By focusing on conceptual understanding and cooperative learning, the model aligns with research-supported educational practices and caters to the varied needs of learners. Successfully teaching quadratic functions necessitates a transition from a teacher-centered approach to a more interactive, student-centered pedagogy that enables all learners to understand and apply essential mathematical concepts with confidence.

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Deep Science Publishing
https://doi.org/10.70593/978-93-49910-85-0

Chapter 5: Students' Errors and Misconceptions in Quadratic Equations

1 Introduction

Quadratic equations are one of the crucial topics, not only in secondary mathematics curriculum around the world but also in the historical development of algebra. The teaching and learning of quadratic equations are presented through factorization, the quadratic formula and completing the square by using representative procedures. Factorization method is the most chosen one by most learners. With this method, learners can solve the quadratic equations rapidly without paying attention to their structure and abstract meaning (Tendere & Mutambara, 2020; Makgakga, 2013). For many secondary school learners, solving quadratic equations is one of the most difficult topics in the curriculum (Tendere & Mutambara, 2020; Makgakga, 2013). It is significant to study the errors, misconceptions and challenges learners have so as to improve their performance in mathematics.

According to Brodie (2014) it is important to identify mathematical problems and determine the areas of weaknesses students make as well as making an effort to explain why those errors are being made. In most cases teachers identify students' errors but hardly analyse them (Luneta & Makonye, 2012). This will enable the teachers to identify the root cause of those errors and how best they can be modified in order to benefit both learners and teachers.

Errors are not simply the result of lack of knowledge or lack of attention but are a result of weaknesses in understanding (Parent, 2015).Learners' misconceptions about quadratic functions have also worried. Parent (2015) who observed that learners tend to think about isolated parts of the problem when solving quadratic problems and relied much on procedural strategies. Students prefer to work with the standard form rather than the vertex form when solving problems on quadratics and also preferred to algebraically solve a problem versus tabular or graphical strategies. According to

Parent (2015) learners may not have a profound understanding of graphs. Learners' errors and misconceptions and choice of strategies in solving quadratic equations will be examined and teaching strategies that will help them deal with the errors and the misconceptions can be developed by engaging on a study of learners' misconceptions. Justification of which points to emphasize in teaching will be sought and this helps in future curriculum planning as suggested in similar studies by Parent (2015). Makonye & Nhlahla (2014) observed that errors and misconceptions in quadratic equations are due to inappropriate schema to solve problems as learners held on to simple equations schema. Their plan should be assimilated to solve quadratic equation. This chapter intends to answer the following questions:

a) What are students' errors and misconceptions in solving quadratic equations?

b) What are the causes of students' errors and misconceptions in solving quadratic equations?

c) What framework can be developed to minimise errors and misconceptionsin solving quadratic equations?

2 Theoretical framework

The cultural environment and the state of society in which learners live all influence how knowledge is created. The idea of a plan was first proposed by Piaget, who said that it aids people in appreciating their surroundings. Assimilation is the process of absorbing new knowledge and incorporating it into the pre-existing strategy. By nature, knowledge construction is more abstract and comes from solutions to issues that people generate sensibly rather than ones that are imposed upon them (Makonye 2010). Learners' difficulties or misconceptions result from the learning process (Makonye, 2010). Peterson (2009) proposed the core concepts of constructivist education; (a) The student actively creates knowledge rather than passively absorbing it from others. It is something that the student does, not something that is forced upon them. (b) Students have preconceived notions about sensations when they enter the learning environment. While some of these concepts are well-developed and firmly established, others are ad hoc and out of balance. (c) Knowledge can be described in some depth and is characterized by the abstract creations of the brain. d) If students' preexisting ideas are to be altered or questioned, teaching must take them very seriously. (e) Although knowledge in one sense is individualized and personal, learners conceptualize their knowledge through their interactions with the physical world, as well as through working together in social settings and in a verbal and cultural context.

Vygotsky provided a psychological perspective on the connection between learning and development.He proposed what he called the Zone of Proximal Development (ZPD), which he described as the gap between what a person can accomplish on their own and what they are unable to accomplish even in the absence of a facilitator.In order to explain learning, the theory also incorporates the ideas of tools, indicators, mediators, and scaffolding.Vygotsky (1978) asserts that mediators might take the shape of a parent, teacher, or learner's experienced person.Mediators employ resources like teaching and learning languages or scaffolding symbols.In order to introduce successful learning, scaffolding is the process of supporting and releasing the social aspect of participatory teaching and learning that occurs inside the Zone of Proximal Development (ZPD).A facilitator must take a student from the exterior ring in the diagram above to the outermost ring in order for them to develop knowledge(Lea & Nicoll, 2013).A degree of uncertainty may be represented by the ring that sits between the innermost and outermost rings (Brodie, 2010).A student in this stage is caught in a misunderstanding, yet in order to advance to the next stage of development while remaining in the Zone of Proximal Development (ZPD), they require a facilitator or someone with greater expertise.The interaction between students and mathematical activities is where learning initially takes place, and then proceed to higher learning levels.

3 Mathematical Errors and Misconceptions

The presence of misconceptions and errorsin learners' early learning makes it difficult for them to manage future demands of mathematics, hence affect their performance in tests or assessment tasks(Tendere & Mutambara, 2020). It is therefore important for teachers need to be 'made' aware of how suchmisconceptions and errors come about and therefore device pedagogical ways of dealing with them. There is a need to develop problem-solving skills so as to be able to deal with misconceptions and errors from learners' daily class activities Berger (2010). Misconceptions and errors are methods for constructing knowledge that must not be eliminated, but rather be capitalised on and used as 'facilitators for inquiry' (Makonye, 2011). From the constructivism view students are not passive receivers of forced facts and information/opinions, but are active participants in the construction of their own knowledge (Clark, 2012). As students participate actively they acquire new knowledge and hence, misconceptions are likely to come from such processes as by-products and errors which are determined and strong to change.

It becomes difficult for teachers to unlearn what the wrong mathematical conceptions (Brodie & Berger, 2010). The thinking abilities of learners' can be recognised from the mathematical conversations they are involved in. From such conversations teachers can pick up misconceptions and errorsand use them to facilitate the learners' process of constructing knowledge (Brodie, 2012). Hearing and listening are also crucial in mathematics teaching and learning process as part of the teacher's role.

According to Makonye (2016) the accuracy of changed knowledge is negotiated by possible uncertainty and different interpretations by diverse people. This happens when information is received by active humans, it gets inferred and enhanced, or rather supplemented, which result with a newly constructed knowledge (Makonye, 2016). The process of enrichment leads to reorganizing and reconstruction of knowledge. When a human being increases more insight in a specific concept this process of enrichment leads to reorganization and reconstruction of knowledge.However, it is unfortunate that to a certain level, what is received does not always remain the same. Such processes of knowledge construction and restructuring are likely to result in misconceptions which lead to making errors. Errors are defined as the systemic wrong answers which come from underlying theoretical structures (Zakaria& Maat, 2010)

According to Makonye and Matuku (2016) a schema is knowledge planned into structures which are large units of reliable concepts. Makonye (2013) defined a perception image as a cognitive portion of ideas that learners form in their minds regardingall aspects of precise concepts which are similar to schema (Makonye, 2013). Whilst the APOS theory suggest that misconceptions and errorsare as a result of failed attempt to adjust or accommodate new ideas, this theory of concept imaging and definition suggest that misconceptions and errorsare as a result of concept images developed by a learners being in conflict with what is believed in and recognized by a wider mathematical community. Likewise, a concept definition which is prone to be in conflict cognitively with a different concept definition is a potential conflict factor and might result in misconceptions.

4 Categories of errors and misconceptions

There are various categories of misconceptions and errors in mathematics education (Luneta & Makonye, 2010).

Random errors

Luneta & Makonye (2010) refer to random errors as lapses or unplanned mistakes. Such misconceptions and errorsdo not have any noticeable or cognitive mathematical reference.

Generalisation

Mathematics learning involves some form of generalisation (Luneta & Makonye 2010), it so happens that learners over-generalise. Over-generalisation happens in at least two ways and these would be over numbers and over operations. Generalisation over number (and number properties) is viewed as the deep level technique from the two levels which guide cognitive operative (Makonye, 2012). An example of this form of over-generalisation could come from a situation whereby learners are asked to find a solution to an equation of the form $x \times y = 0$. Definitely with this form of an equation it does allow for one to continue by saying $x = 0$ or $y = 0$, due to a property of a zero as different to that of any other number. The facts that a quadratic equation which factories to $(y - 3)(y + 5) = 0$ would produce two linear equations $y - 3 = 0$ and $y + 5 = 0$ may be generalised over to a state whereby the right hand side of the equation is not a zero. Learners may assume that because the above is true and mathematically reasonable, another equation which may look as $(y - 1)(y + 4) = 6$ for example should equally work out to $y - 1 = 6$ or $y + 4 = 6$ inaccurately resulting with $y = 7$ or $y = 2$. This is an example of generalisation which disrespects the difference in properties of numbers.

Generalisation over operations

It may be seen at a step when a negative number is presented. For example with a correct statement $(+5) + (+3) = (+8)$, which tolerates no changed answer to $(+3) + (+5)$ due to addition taking a commutative property, a learner may incorrectly consider that $(+8) - (+5) = (+5) (+8)$. This would be captivating the commutative property of the adding operation and over-generalising it to the subtraction operation which may also apply by considering the subtraction operation before the 5 as removed from the number. The error is as a result of the comfort learners derive when writing a positive number with no plus sign in front of it. Learners have a habit of allocating the same resolution to the process of working with negative numbers, in which it put on a little different, therefore $(+8 - (+5)$ is erroneously taken as calculating $(+ 5) - (+8)$.

Interference

According to Luneta and Makonye (2013) errors and misconceptions are difficult to remove due to the existence of knowledge learnt at an earlier stage. When new knowledge is introduced, it has to be attuned into the present plan. For example, learners may take 3^2 which is equal to 3×3 which is equal to 9, and rewrite it as $3 \times 2 = 6$ due to the frustration which may come from interference.

Ignorance of rule restriction

Taking rules which were appropriate in one domain and force them into another domain result in ignorance of the restriction. An example of this may be found from ordering of decimals (Brodie, 2011). The knowledge that the more the digits in a number, the bigger the number is a conception in the whole number domain, but a misconception in a fractional domain, and produces an error that: 0,245 > 0,5.

Incomplete application of rule

This would be seen from where a learner applies a rule correctly, and then not be able to proceed to the next stage of the solution.

5.Results

The details of the errors made by the learners classified according to their type are shown in the table 5.1.

Table 5.1 Test results for 20 participants.

Question number	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**
Number of learners passed	15	12	7	10	10	13	11	9	14	12
Percentage % pass rate	75	60	35	50	50	65	55	45	70	60
Generalisation over operations	2	5	2	10	5	3	1	0	0	4
Generalisation over numbers	9	3	9	3	6	8	4	7	5	10
Random errors	0	0	6	0	9	2	0	1	3	2
Ignorance of rule restriction	0	0	8	4	0	5	6	0	7	0
Interference	0	0	0	0	0	0	3	9	5	0

Question 1 was based on factorised equations and was equated to zero. Table 5.1 indicates that they are a total of 2 generalisations over operations and 9 generalisation over numbers that were made by the learners made. The common type of errors made by learners was generalisation over numbers and followed by generalisation over operations. The example of the learner's respond above is that the learner failed to realise that the equation is given in factors form, instead the learner multiplied out the brackets correctly which was unnecessary and even crosses x^2 the equal sign and then divide both side by x. 15% of the learners failed to answer the question and 85% of the learners have master the concept, a few learners failed to come up with the values of the unknown. There was very high performance of learners on this question and it shows that the learners master the concept of this question very well.

Question 2 was also based on factorised equations and was equated to zero. Table 1 above indicates that they are a total of 3 generalisation over numbers followed by 5 generalisation over operations which was made by the learners. The learners have 40%

of the errors whereby few of them failed to find the values of the unknown and 60% of the learners were able to solve the equation. The learners' performance in question 2 indicates that they have mastered the concept of this question very well and the performance was high.

Question 3 was based on factorised equations and was equated to 6. Table 1 indicates that they are a total of 2 generalisation over operations, 9 generalisation over numbers, 6 random errors and 8 ignorance of rule restrictions made by learners. 65 % of the learners have made errors and only 35% passed the test, this question consequently revealed that learners experience problems in solving quadratic equation. There was low performance of learners on this question and it shows a lack of concept mastering.

Question 4 above was based on the equation which was not factorised and 50% of the learners failed the question and have made errors when dealing with operations on directed numbers and 50% passed this question. Table 5.1 shows that there are total of 10 generalisation over operations, 3 generalisation over numbers and 4 ignorance of rule restrictions. The most common type of error made by learners were generalisation over operations, ignorance of rule restrictions and the least were generalisation over numbers.

Question 5 was based on the equation which was not factorised and 50% of learners made errors. There are a total of 5 generalisation over operations, 6 generalisation over numbers and lastly 9 random errors (see table 5.1). the common type of errors made by learners were random errors followed by generalisation over numbers and lastly generalisation over operations. Random errors were also made in solving quadratic equations (see table 5.1).

Question 6 was based on the equation which was factorised with a power of 2 and equated to zero. The table 5.1 shows that there are a total of 3 generalisation over operations, 8 generalisation over numbers, 2 random errors and 5 ignorance of rule restrictions. The most common type of errors made by learners were random errors followed by generalisation over numbers, ignorance of rule restrictions, generalisation over operations and lastly random errors. 60% of learners failed to solve the quadratic equations which are equated to any number which is not zero 40% passed. The learners have low performance in this question. Learners did not master the concept of this question very well. Solving this question was based on the reasoning that if a and $b = 0$ then either a or b is0, or both $a$and$b$ are equal to zero.

Question 7 was based on the equation which was not factorised and equated to 16. There are a total of 1 generalisation over operations, 4 generalisation over numbers, 6 ignorance of rule restrictions and lastly 3 interference (see table 5.1). Most type of errors made by learners was ignorance of rule restrictions followed by generalisation over numbers, interference, and lastly generalisation over operations. 45% of learners failed to solve the equation. The learners performed low in this question.

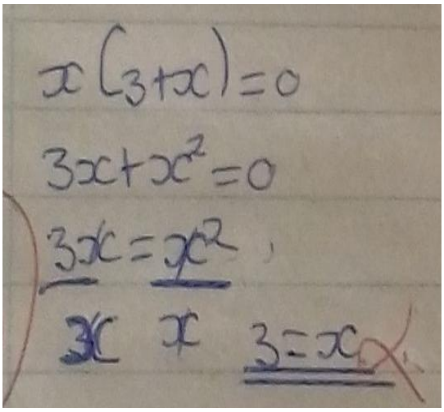
Question 8 was based on the equation which was not factorised and equated to zero. The table 5.1 shows that there are a total of 9 interference, 7 generalisation over numbers and 1 random error. Most type of errors made by learners were interference followed by generalisation over numbers and lastly random errors. 65% of them were failing to find the difference of two squares and 35 % of the learners passed the question 8. The learners have low performance in this question. Learners did not master the concept of this question very well. Solving this question was based on the reasoning that if $a$and$b = 0$ then either a or $b$is 0, or both $a$and$b$ are equal to zero. This indicates to the fact that the students may not have used the algebraic methods or they may have difficulties in applying algebraic methods to solve equation.

Question 9 was based on the equation which was not factorised and equated to 16. The table 5.1 shows that there are a total of 7 ignorance of rule restrictions, 5 generalisation over numbers, 5 interference and 3 random error. Most type of errors made by learner's ignorance of rule restrictions followed by generalisation over numbers, interference and lastly random errors. 65% of learners have errors in solving this quadratic equation and 35% passed question. The learners have low performance in this question. Learners did not master the concept of this question very well. Solving this question was based on the reasoning that if a and $b = 0$ then either a or b is0, or both$a$and$b$ are equal to zero.

Question 10 was based on the concept of word problems. The table 5.1 shows that there are a total of 4 generalisation over operations, 10 generalisation over numbers, 2 random errors. Most type of errors made by learners were generalisation over numbers followed by generalisation over operations and lastly random errors. The learners were required to solve and understand the word problems applying the skills they had learnt in class. 60% of the learners failed to solve word problems and 35% of them were able to solve question 8. There was a very low performance in learners in this question.

Generalisation over numbers

Twelve learners made generalisation over numbers error in question 1 and 9. Learners also made the same error in question 3. The researcher interviewed the learners according to their answers. Figure 5.1 and figure 5.2 are a case of Generalisation over numbers for Loo and Fifi respectively.



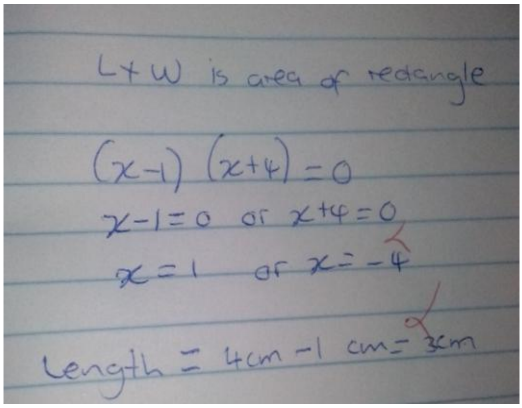


Fig 5.1: Fifi' answer **Fig 5. 2: Loo's answer**

<u>Figure 5.1 Fifi's case</u>

Interviewer: why did you remove the brackets?

Fifi: because we learnt that if we have a bracket, we should remove bracket by multiplying it with the algebraic term outside the bracket.

Interviewer: Oooh, Okay so why did you divide with$x$ both sides?

Fifi: So that it should be left with the subject of the formula x and a value which says $x = 3$.

Interviewer: okay.

In the analysis of the interview above Fifi did not understand the difference between x^2 and x the learner also did not master the concept of zero product. The rule applies in the case whereby the one side of the equation is equated to 0, for example $(3 + x) = 0$ and$x = 0$. Then the rule would apply. This was supported by constructivism view of learning and knowledge formation, which must viewed as a normal process of learning as it enables the construction of new knowledge (Makonye, 2013).

Another error I picked up on Loo's solution in figure 5. 2 above. In his response the learner justified the solution as follows:

<u>Loo's case</u>

Interviewer: Why do you equate to zero?

Loo's respond: Because the quadratic formula is equated to zero.

In the analysis of the interview above the learners knows that the quadratic formula is equated to zero and it clearly indicates that the learners were in a position to know how to find the area of the rectangle by multiplying $(x - 1)$and$(x + 4)$. The product of

$(x - 1)$ and $(x + 4)$ was then equated to zero instead of the given area 6cm². The interview data showed that formulating a quadratic equation was quite challenging. The misunderstanding of the teacher's instruction on equalization of zero and given area appears to be the cause of this error. The constructivist perspective was that the mistakes made by students are characterized by their inability to translate word problems into equations, lack of comprehension, and basic skills that should have been learned in lower grades. Brodie & Berger (2010) reported that teachers struggle to convince students that their knowledge is not accurate mathematical concepts. The cognitive abilities of learners can be identified through their participation in mathematical discussions.

Generalisation over operations

Ten learners made this error and the researcher interviewed the learners according to their answers. Figure 5.3 and 5.4 are a case of generalisation over operations for Bongi and Tawa respectively.

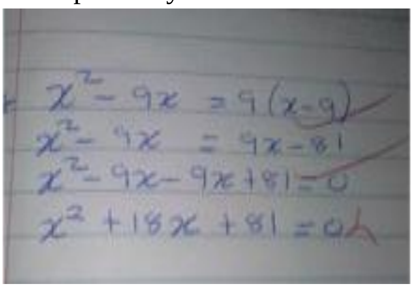


Figure 5.3: Bongi

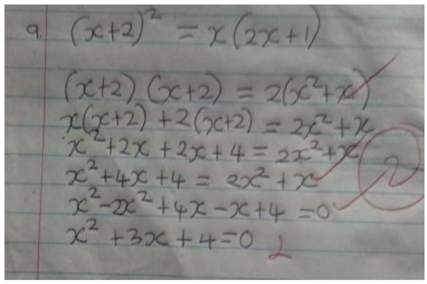


Fig 5.4: Tawa

The researcher obtained the following responses from the carried interviews.

<u>Bongi's case in Figure 5. 3</u>

Interviewer: why did you put a$+18x$?

Bongi: when we were given like signs we just add the numbers such as $+9x\ +9x$.

Interviewer: okay so you mean when we add $+9x\ +9x$ and$-9x - 9x$, do we get the same answer?

Bongi: Oh sorry ma'am, when we have sum of algebraic terms which have like signs we add the algebraic terms and take the common sign so it becomes$-18x$.

Another error I picked up on Tawa's solution in figure 5.4 above. In his response the learner justified the solution as follows:

<u>Case of Tawa</u>

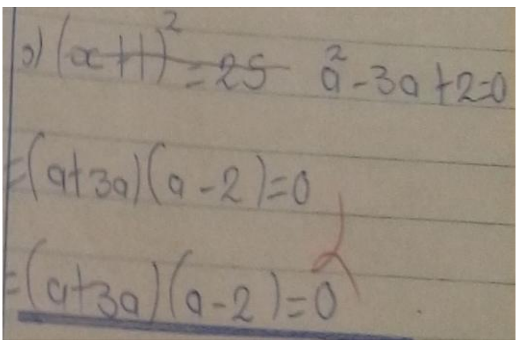
Interviewer: Tawa, look at the last stage of your solution.

Tawa: Okay ma'am.

Interviewer: As you can see, other stages are correct. What happened to the last stage?
Tawa: I subtract x^2 - $2x^2$ and i got x^2
Interviewer: can we subtract 2 items from 1 item.
Tawa: No ma'am.
Interviewer: okay so how come you subtract x^2 from $2x^2$ and got x^2.
Tawa: I don’t know the solution ma’am.

During the interview analysis, Bongi proposed simplifying problems by using like signs on directed numbers and failed to follow the correct rule. Due to the existence of similar signs, Bongi must add numerals and choose from the ones outlined. The incorrect answer by Tawa, indicates his lack of knowledge about operations. He should have chosen to subtract the largest number from its smaller one and then signify the larger number. Makonye (2012) posited that the absence of abstract knowledge and the inability to link new information with old information are the reasons for such errors. This is supported by this theory. During the combination of numbers, students do not manipulate the plus or minus signs in front of them. It was observed by the researcher that learners should slow down when writing and increase their practice to prevent panicking during exams. **Random errors**

Nine learners made this error and the researcher interviewed the learners according to their figures. For example, figure 5.5 and 5.6 are a case of random errors for Talia and Fay respectively.



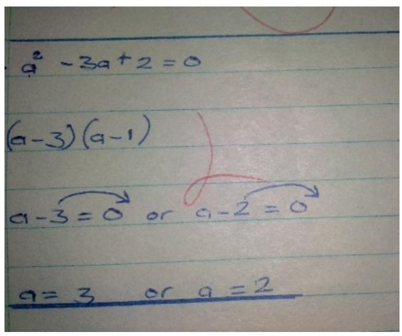


Fig5. 5: Talia   **Fig 5. 6: Fay**

<u>Case of Talia fig 5.5</u>

Interviewer: Please can you read question in your answer sheet.

Talia: I am done ma’am

Interviewer: okay. Why did you choose $+2$ and $+1$ as your factors?

Talia: Because their product gives $+2$

In the analysis from the above interview, Talia has an idea that the product of the first and last term is $+2$ and however, an incorrect method was used to solve the quadratic equation. Hence, the researcher advised the learner to take note of the sum of two factors so as to match the middle term of the quadratic equation. The constructivist theory of learning suggest that learners come to a new grade not as empty vessels but they come with some pre-knowledge developed in the previous grades and this knowledge was used to adjust and get used to incoming mathematical concepts (Zakaria & Maat, 2010).

<u>Talia's view</u>

Talia said that '*i did not understand this topic very well so I will talk to my friends so that we form a group and help each other'*.

Another error I picked up on Fay's solution in figure 6 above. In her response the learner justified the solution as follows:

<u>Case of Fay fig 5.6</u>

Interviewer: Read the question in your answer sheet and tell me how you solve it.

Fay: I dont know how to work this question so I was just trying.

Interviwer: Okay.

In the analysis of the interview above Fay has no idea on solving quadratic equations. As per the constructivist viewpoint, knowledge cannot be transferred from one person to another, but rather is constructed by an individual who actively participates in the process (Brodie 2010). Learning can be directly involved in the process of constructing knowledge, by making learners aware of their own mistakes. In order to effectively teach and learn these concepts, it is essential to adopt a student-centered approach.

Ignorance of rule restrictions

Seven learners made this error and the researcher interviewed one of the learners. For example, figure 5.7 and 5.8 are a case of ignorance of rule restrictions for Thandie and Manex respectively.

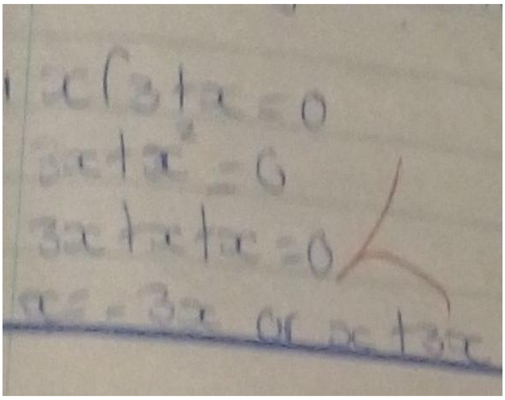


Fig 5.7:Thandie

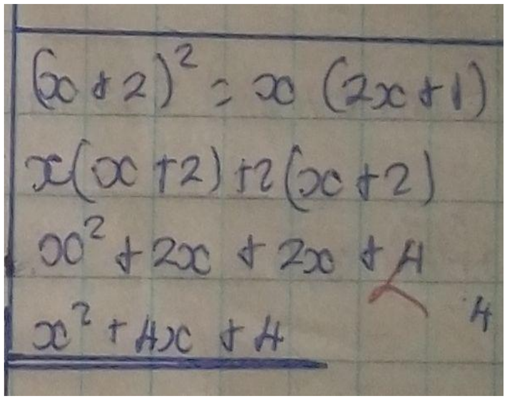


Fig 5. 8: Manex

The researcher obtained the following responses from the carried interview.

Case of Manex figure 5.7

Interviewer: Manex, please let's have a look at line four.

Manex: Okay ma'am

Interviewer: How did you get $3x^2$?

Manex:I added $2x^2 + x$

Interviewer:Okay. Is it possible to mix cows and hyenas in one place?

Manex: No ma'am, it's impossible.

Interviewer: Okay good boy, so we cannot add $2x^2$ and x also.

Manex: why ma'am?

Interviewer: because they are unlike terms.

In the analysis of the interview above Manex has no idea of unlike terms, he expands the equation correctly and then failed to simplify algebraic terms with unlike terms. Peterson (2010) noted that even though knowledge in one sense is personal and individual, the learners' understand concepts through their associations with the physical world, collaboratively in societal situations and in cultural and oral environment. This needs to be improved by attaining knowledge of quadratic rules.

Another error I picked up on Thandie's solution in figure 5.8 above. In her response the learner justified the solution as follows:

Case of Thandie figure 5. 8

Interviewer: Please read the question.

Thandie: I am done.

Interviewer: look at line three. Where did you get $x + x$ from?

Thandiwe: because we have x^2 so my teacher said if we have a letter with a power squared on top it means we have two letters, so that's why I add x and x.

Interviewer: Okay. So if you add $x + x$ you get $2x$

Thandie: Yes ma'am.
Interviewer: look at the last line. Where did you get $x - 3x$ *or* $x + 3x$ from?
Thandiwe: I don't know ma'am.
In the analysis of the interview above Thandie had no idea of solving the quadratic equations using the zero product so the learner tried to ignore $x^2$and going down with the solution, the learner indeed continued working out the sum and just put $x - 3x$ and $x + 3x$. The new knowledge is occasionally in conflict with what already present in an individual's mind (Brodie, 2012). More practice is needed by Thandie so that she grasp the concepts of quadratic equations and also remedial is needed.

Interference

Fourteen learners made this error and the researcher interviewed one of the learners. For example, figure 5.9 and figure 5.10 are case of Interference for Betty and Leo respectively.

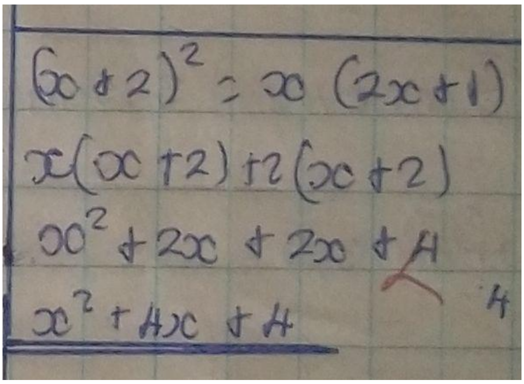


Figure 5.9: betty

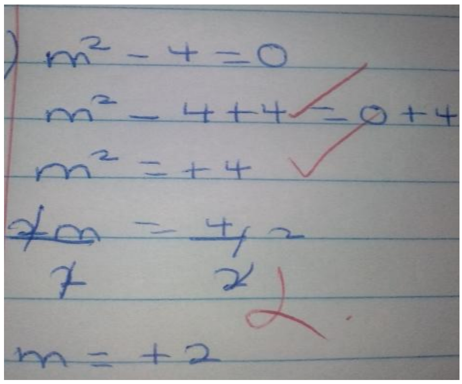


Figure 5.10: leo

<u>Betty's case figure 5. 9</u>
Interviewer: Can you please go to question 8 .Read the question and the solution that you provided for that question.
Betty: I am through.
Interviewer: Explain to me how you came up with the answer 2.
Betty: I add 4 both sides and divide with 2 to get +2
Interviewer: are you saying m^2 is the same as 2m.
Betty: Yes ma'am.
In the examination of this response from the learner, it is evident that the squaring concept was not well-defined. The answer she provided was a result of chance rather than the usual mathematical procedures. One of the mistakes made according to by

Leslie & Nicoll (2013) was that learners often pursue an incorrect idea and present it as though it were correct out of desperation.

In the figure 10 above, I discovered a mistake in Leo's solution. In his response the learner justified the solution.

Leo’s case figure 5.10
Interviewer: Leo, let’s talk about your solution to figure 10. Please have a look at it.
Leo: Okay ma’am.
Interviewer: Okay, what happened to the equal sign?
Leo: Where ma’am?
Interviewer: On line two. Oh, and line three. The equal sign seems to have disappeared there.
Leo:Uhm…It was a mistake ma’am.
Interviewer: Okay. Can you tell me what mistake it was?
Leo: I expand $(x+2)^2$ and forgot to write the equal sign.
Interviewer: okay.
In the analysis of the interview above the learner just forgot to put an equal sign. Makonye (2013) supported that learners mistakes might be because of environment or personal issues.

Causes Of Errors In Solving Quadratic Equation

Teachers were asked by the researcher on factors causing errors in solving quadratic equations. T1 stated that “*the use of English language in the learning of mathematics is a major cause of errors and this was proved by the majority of learners understanding Shona language than foreign language”*. According to Brodie (2012), students' lack of proficiency in the English language, which is primarily used during teaching, constitutes a significant error. This view is supported by other studies. Consequently, the epistemic access of quadratic equation knowledge was compromised for learners who had to convert their language of teaching and learning to English in order to obtain mathematic epistemic access. Mathematics is a language in its own right (Zakaria& Maat, 2010). Learning mathematics required overcoming both English and mathematical barriers. Both languages were problematic for learners. The importance of language in the process of learning was highlighted by Zakaria and Maat (2010).

T 2 stated that “*learners usually make errors in learning quadratic equations because it is inside them, most learners usually make mistakes and as such they cannot fully complete a test without making a mistake”*. As a result, it is evident from the statement above that some students have adapted to making mistakes.

The statement made by Makgakga (2013) is that errors are habitual and require a profound solution. The interviews that were conducted with learners after they completed a quadratic equations test demonstrated this.' According to Makgakga (2013), learners' errors occur frequently and with great frequency. According to Makgakga (2013), it is important for teachers and learners to discuss errors during teaching and learning so that they can identify areas where they may be eliminated. T3 suggests that "signs *especially when the brackets come one differently after factorisation".*Some researchers suggest that errors arise from prior knowledge as learners attempt to construct mathematical knowledge meanings (Luneta & Makonye, 2012). Learning difficulties arise when attempting to make mathematical sense. The excessive intervention of the learners, other learners and their teachers, as well as the surrounding environment, leads to confusion.

On the other hand, learner's errors were attributed to lack of arithmetic skills. Learners seemed to lack good arithmetic skills. Learners committed errors because of poor arithmetic background. This is supported by Makonye (2011), who also specified that poor arithmetic skills contribute to errors. They are other factors which include unavailability of teaching resources such as textbooks, teaching method used, the school environment and attitude towards mathematics. One of the teacher remarked," *Learners are failing to cope the concept of quadratic equations because they lack practice*" (T4). Lack of practice was discovered also as a factor of errors especially at home which are not also mounted on the boy child. A lack of understanding of the concepts such as directed numbers, operation of numbers was also noted..

T3 claimed that*" the learners should be of mixed abilities that are also the factor of errors made by learners*". Students are better classified into different classes when it comes to the incidence of errors. Assessing students in accordance with their teachers' perceptions of competence is the most significant labelling, which impacts both performance and errors. Students in a substandard class tend to perform worse than those who are identified as gifted. Classes must have mixed ability, so that weaker learners can interact with fast learners and gain support from other classmates.

T4 said that *"learners are just reading questions without understanding them*". According to the researcher, learners must read questions slowly and practice more frequently to prevent panic during tests. Those who were converting equations into quadratic expressions were particularly affected by this, as they were only able to factorize and not derive their roots. It was suggested by Mbewe (2011) that students who exclusively used factorization and relied on the quadratic formula may have developed a negative bias towards the factorizing method, leading them to study it instead. Other factors may contribute to the wrong method's use, as they are not identified as appropriate. The lack of comprehension of directed numbers among students may be the cause of the switchover between indicators.L2 said that "*quadratic* *expressions and factorisation are my problems; I did not master these topics because our teacher did not explain clearly about these topics.* The learners did not master these topics because their teachers are not explaining very well to them. T5 suggested that '*I think teachers are not sequencing and selecting topics very well'*

Strategies of minimising errors

The respondents were asked to make suggestions on how errors the students made could be reduced or eliminated. T2 suggests that "*encourage able students to share their computation strategies with the whole class".*

T1 "*there is need to apply different teaching methods to enhance learners understanding and to improve learner's motivation with respects to mathematics since most learners develop negative interest in mathematics*," the teacher suggested that teachers should use group work as the most effective teaching methods. This suggests that the teaching approach or method used by teacher has a great effect on improvising or motivating the learners to perform better or to develop a positive attitude towards the subject. This agrees with Makonye and Matuku (2016) who stated that problem solving using ill-structured problems and group discussions motivates students and encourages understanding the epistemology of the discipline.

T3 was of the view that learners should develop critical thinking, the teacher stated that, "*learners should be given the opportunity to reason and solve problems without the help of teachers, and thus primary education should lay the foundation in relation to solving quadratic equations".* This clearly shows that, the teacher has a great impact on student's attitude towards mathematics and also considerably affect student`s performance in the subject area. Therefore, attitude towards mathematics denotes interest or feeling. It is the students' character towards like 'or dislike 'in maths.

Other teachers were of the view that, there is need to have experienced teachers particularly in the epistemological errors to enhance learner's performance in the subject. Experience has a positive impact on the learner's performance (Didis & Erbas, 2015). Therefore, teachers with applicable experience are able to adapt their learning in respect of the learner's challenges. However, this kind of experience would have been excellent if the curriculum had stayed the same during all their years of teaching experience. T5 mentioned that, "*learners lack mastery of directed numbers and also use of wrong rules or strategies".* This clearly shows that these factors result in learners not being mathematically skilful to solve the quadratic equations. This is supported by the amount of different types of errors displayed on their scripts.

All teachers disputed the use of calculators in solving mathematical problems at form one level since this affect the learners thinking capacity and will negatively affect understanding of quadratic equations for example, T4 mentioned that,” *when a learner continues to use electronic devices to calculate simple mathematical problems, he or she will damage his or her mind and this will even affect such learners in the future”*. Furthermore, since the learners had no understanding, they had no idea of how to check whether their answers were right or wrong if, for example, they may punch the buttons wrongly. Use of calculators must be deferred until learners have developed relational understanding of quadratic equations, although they can still be used intelligently in an examining sense. Therefore, knowledge on quadratic equations is gained over a long period of time.

L1 suggests that “*the teacher should teach us directed numbers and quadratic expression because we did not master these topics very well*”. Critical topics that have a bearing on the generation of errors when quadratic equations should be thoroughly and deeply looked at before quadratic equations are covered. This would minimize the guess work that Didis and Erbas (2015) highlighted as one cause of the errors. Their sequencing and the content to be included in each topic needs to be carefully done so as to enhance the performance of learners.They stated that the textbook should be provided for teachers and learners. One of the learners said that*” the school should provide more text books for us so that each one of us will have his or her book*”. (L3). Constructivism theory supported that misconceptions must be exploited by a teacher as opportunities to enhance learning and knowledge construction. .

T5 claims that ‘teachers *should always analyse learners work*’ and it is supported by Makonye and Khanyile (2015) who reported that the only way teachers can access their learners’ thinking is through error analysis.

6 Constructivist Remediation and Progression framework for error and misconception identification

The Constructivist Remediation and Progression framework (see fig 5.11) is based on the findings.The purpose of this framework is to facilitate the detection, examination, and correction of learner errors pertaining to the generalization over numbers in quadratic equation solutions. These mistakes usually occur when students misuse algebraic rules, including using the zero-product property in the wrong situations. For example, students may incorrectly equate each element to 6 by extrapolating the proper method for calculating an equation such as $(x-3)(x+5)=0$ and $(x-3)(x+5)=0$ to a problem such as $(x-1)(x+4)=6$ and $(x-1)(x+4)=6$. These misunderstandings show a lack of knowledge about the application of particular rules or features.

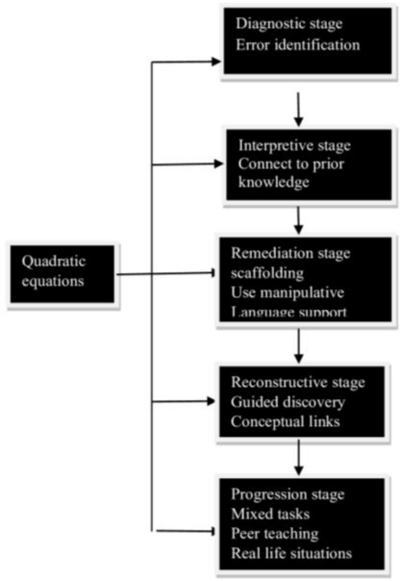


Fig 5.11 Constructivist Remediation and Progression framework

The diagnostic stage, the first part of the architecture, seeks to determine and categorize the particular kind of mistake the learner is making. To determine if students comprehend the circumstance in which the zero-product property is applicable, that is, when one side of the equation equals zero teachers should employ a combination of written diagnostic tests and oral interviews. This offers vital information on the conceptual framework of the learner

The interpretative stageuses constructivist ideas to identify the error's causes. According to this viewpoint, students expand on what they already know, which can occasionally result in overgeneralization. These mistakes should be viewed as a normal part of the process of creating knowledge rather than as failures. The persistence of these mistakes could be a sign that the student is struggling with the application and timing of particular rules and is in a transitional stage of conceptual growth. Teachers are urged to acknowledge that a key factor in deeper learning is cognitive conflict, which occurs when students realize that their present thinking is not producing the right answers.

The focused educational interventions are the main emphasis of the remediation stage. These ought to contrast legitimate and illegitimate uses of the zero-product rule using a variety of representations. Teachers could, for instance, use an area model to visually illustrate the meaning of $(x-1)(x+4)=6$; $(x-1)(x+4)=6$ and contrast it with $(x-1)(x+4)=0$; $(x-1)(x+4)=0$. Incorporating error analysis exercises into the classroom, where students review and fix flawed solutions, can also assist students in understanding the reasons behind the failure of particular tactics. Additionally, learners should be moved from simple circumstances (like RHS = 0) to more difficult contexts (like RHS = a non-zero constant) using scaffolding. In order to guarantee conceptual clarity, teachers should incorporate multilingual instruction for students who are more proficient in their native tongue, such as Shona.

The reconstructive stage, the fourth element, is where students reassemble accurate mathematical knowledge. This can be accomplished through guided discovery learning, in which students are not taught mathematical rules but are instead guided to rediscover them. Instructors ought to urge students to describe the circumstances and reasons behind the zero-product property's operation. Students may also maintain reflective notebooks in which they record their thought processes and consider when particular practices are appropriate or inappropriate. It is easier to consolidate precise generalizations with arithmetic operations (e.g., knowing that only $3\times 0 = 0$, not $3\times 2 = 0$) and algebraic principles are explicitly connected.

The progression stage is intended to assist students in advancing after they have attained conceptual comprehension. This entails making use of Vygotsky's Zone of Proximal Development (ZPD) by establishing chances for teacher scaffolding and peer tutoring. To enhance comprehension, practice problems should have a variety of formats, such as situations in which the equation is not set to zero and in which the right-hand side has an additional expression. To help students understand the importance and significance of the mathematics they are studying, teachers should also relate algebraic ideas to practical applications, such as modeling area or motion with quadratic equations. Through this process, students are more likely to build a strong mental schema and are less likely to rely on processes they have memorized.

The Constructivist Remediation and Progression Framework helps students transition from overgeneralization to precise algebraic principle application. It accomplishes this by identifying misunderstandings, analyzing them from a constructivist perspective, addressing them with focused tactics, reassembling accurate information, and promoting advancement with supervised assistance. This approach, which is based on both constructivist and APOS theories, acknowledges that mistakes like overgeneralizing about numbers are developmental stages that, with the correct pedagogical assistance, can be converted into long-lasting mathematical comprehension.

Conclusion

With an emphasis on the most prevalent error types found through diagnostic tests and interview data, this chapter has examined the nature, causes, and consequences of learners' mistakes and misconceptions when solving quadratic equations. According to the analysis, a large number of these errors are caused by the overgeneralization of mathematical rules, specifically the misuse of the zero-product property; a lack of knowledge about operational properties; haphazard procedural errors; a lack of awareness of the limitations of algebraic rules; and interference from preconceived notions. These results highlight how crucial it is to view learner errors as significant markers of underlying cognitive structures and developmental phases rather than just as failures.

The conversation, which was based on constructivist and APOS learning theories, emphasized that students do not come into class with a clean slate; rather, they bring pre-existing knowledge and intuitive understandings that could be at odds with formal mathematical notions. These assumptions frequently lead to the development of flawed but internally coherent approaches to problem-solving. Language hurdles, a lack of basic mathematics knowledge, a lack of practice, and an inefficient way of organizing the curriculum's content were also found to be major contributors to these mistakes.

With an emphasis on the most prevalent error types found through diagnostic tests and interview data, this chapter has examined the nature, causes, and consequences of learners' mistakes and misconceptions when solving quadratic equations. According to the analysis, a large number of these errors are caused by the overgeneralization of mathematical rules, specifically the misuse of the zero-product property; a lack of knowledge about operational properties; haphazard procedural errors; a lack of awareness of the limitations of algebraic rules; and interference from preconceived notions. These results highlight how crucial it is to view learner errors as significant markers of underlying cognitive structures and developmental phases rather than just as failures.

In order to help instructors identify, evaluate, and correct student faults, the chapter suggested the Constructivist Remediation and Progression Framework (CRPF). Within the learner's Zone of Proximal Development, this paradigm promotes a learner-centered, developmentally appropriate, and conceptually based approach to mathematics instruction that places a strong emphasis on reasoning, reflection, and scaffolding.

The results of this study urge a change in instructional strategies toward approaches that utilize mistakes as a foundation for developing deeper understanding rather than just fixing them. Instructors are urged to use a variety of approaches when tackling quadratic equations, encourage students to discuss misconceptions in the classroom, and create a safe space for them to make and examine mistakes. Meaningful and lasting mathematical understanding can only be attained through such reflective and responsive instruction.

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